(1) $\vec{v_1} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$ $\vec{v_2} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$ $\vec{v_3} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$\vec{v_1} \cdot \vec{v_2} = 0$ $\vec{v_1} \cdot \vec{v_3} = 0$ $\vec{v_2} \cdot \vec{v_3} = 0$

The set is orthogonal.

(3) $\vec{v_1} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$ $\vec{v_2} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ $\vec{v_3} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$

$\vec{v_1} \cdot \vec{v_2} \neq 0$

The set is not orthogonal.

(5) $\vec{v_1} = \begin{bmatrix} 2 \\ 3 \\ -1 \\ 4 \end{bmatrix}$ $\vec{v_2} = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ $\vec{v_3} = \begin{bmatrix} -4 \\ -6 \\ 2 \\ 7 \end{bmatrix}$

$\vec{v_1} \cdot \vec{v_2} = 0$ $\vec{v_1} \cdot \vec{v_3} = 0$ $\vec{v_2} \cdot \vec{v_3} = 0$

The set is orthogonal.

(7) $\vec{v_1} \cdot \vec{v_2} = 0$

2 nonzero orthogonal vectors in $\mathbb{R}^2$ are a basis for $\mathbb{R}^2$

Orthogonal basis $\Rightarrow$ $\vec{w} = \mathrm{proj}_{\vec{v_1}} \vec{w} + \mathrm{proj}_{\vec{v_2}} \vec{w}$

$$= \frac{\vec{v_1} \cdot \vec{w}}{\|\vec{v_1}\|^2} \vec{v_1} + \frac{\vec{v_2} \cdot \vec{w}}{\|\vec{v_2}\|^2} \vec{v_2}$$

$$= \frac{10}{20} \begin{bmatrix} 4 \\ -2 \end{bmatrix} - \frac{5}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 4 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Coordinate vector $[\vec{w}]_\beta = \begin{bmatrix} \frac{1}{2} \\ -1 \end{bmatrix}$

(the coefficients from the linear combination above)

(9) $\vec{v_1} \cdot \vec{v_2} = 0$, $\vec{v_1} \cdot \vec{v_3} = 0$, $\vec{v_2} \cdot \vec{v_3} = 0$

3 nonzero orthogonal vectors in $\mathbb{R}^3$ are a basis for $\mathbb{R}^3$

Orthogonal basis $\Rightarrow$ $\vec{w} = \mathrm{proj}_{\vec{v_1}} \vec{w} + \mathrm{proj}_{\vec{v_2}} \vec{w} + \mathrm{proj}_{\vec{v_3}} \vec{w}$

$$= \frac{0}{2} \vec{v_1} + \frac{4}{6} \vec{v_2} + \frac{1}{3} \vec{v_3}$$

$$= \frac{2}{3} \vec{v_2} + \frac{1}{3} \vec{v_3}$$

Coordinate vector $[\vec{w}]_\beta = \begin{bmatrix} 0 \\ 2/3 \\ 1/3 \end{bmatrix}$

(13) $\vec{v_1} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$ $\vec{v_2} = \begin{bmatrix} 2/3 \\ -1/3 \\ 0 \end{bmatrix}$ $\vec{v_3} = \begin{bmatrix} 1 \\ 2 \\ -5/2 \end{bmatrix}$

$\vec{v_1} \cdot \vec{v_2} = 0$ $\vec{v_1} \cdot \vec{v_3} = 0$ $\vec{v_2} \cdot \vec{v_3} = 0$ ✓

$$\|\vec{v_1}\| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}}$$
$$= 1$$

$$\|\vec{v_2}\| = \sqrt{\frac{4}{9} + \frac{1}{9}}$$
$$= \sqrt{\frac{5}{9}}$$
$$= \frac{\sqrt{5}}{3}$$

$$\|\vec{v_3}\| = \sqrt{1 + 4 + \frac{25}{4}}$$
$$= \sqrt{\frac{45}{4}}$$
$$= \frac{\sqrt{45}}{2}$$
$$= \frac{3\sqrt{5}}{2}$$

$\left\{ \vec{v_1}, \ \frac{3}{\sqrt{5}} \vec{v_2}, \ \frac{2}{3\sqrt{5}} \vec{v_3} \right\}$ is an orthonormal set.

$\left\{ \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 2/3\sqrt{5} \\ 4/3\sqrt{5} \\ -5/3\sqrt{5} \end{bmatrix} \right\}$ is an orthonormal set.

(15) $\vec{v_1} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$ $\vec{v_2} = \begin{bmatrix} 0 \\ \sqrt{6}/3 \\ 1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$ $\vec{v_3} = \begin{bmatrix} \sqrt{3}/2 \\ -\sqrt{3}/6 \\ \sqrt{3}/6 \\ -\sqrt{3}/6 \end{bmatrix}$ $\vec{v_4} = \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\vec{v_1} \cdot \vec{v_2} = 0$ $\vec{v_1} \cdot \vec{v_3} = 0$ $\vec{v_1} \cdot \vec{v_4} = 0$ ✓

$\vec{v_2} \cdot \vec{v_3} = 0$ $\vec{v_2} \cdot \vec{v_4} = 0$ $\vec{v_3} \cdot \vec{v_4} = 0$

$$\|\vec{v_1}\| = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}} = 1$$

$$\|\vec{v_2}\| = \sqrt{\frac{6}{9} + \frac{1}{6} + \frac{1}{6}} = 1$$

$$\|\vec{v_3}\| = \sqrt{\frac{3}{4} + \frac{3}{36} + \frac{3}{36} + \frac{3}{36}} = 1$$

$$\|\vec{v_4}\| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

The original set is orthonormal.

(17) Check if $Q^T Q = I$

$$Q^T Q = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

The matrix is orthogonal.

For any orthogonal matrix Q, $Q^{-1} = Q^T$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

(23) Let Q be an orthogonal matrix.
Show that $\det Q = \pm 1$.

Q orthogonal

$\Rightarrow Q^T Q = I$

$\Rightarrow \det(Q^T Q) = \det I$

$\Rightarrow \det Q^T \cdot \det Q = 1$

$\Rightarrow \det Q \cdot \det Q = 1$

Let $\det Q = x$

$x^2 = 1$

$x = \pm 1$

$\det Q = \pm 1$