(5)

Eigenvalues of $A = 4, 3$ (the diagonal entries of D)

Basis for E_4 = 1st column of P

$$= \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

Basis for E_3 = 2nd column of P

$$= \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

(7) Eigenvalues of $A = 6, -2$ (the diagonal entries of D)

Basis for E_6 = 1st column of P

$$= \left\{ \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \right\}$$

Basis for E_{-2} = last two columns of P

$$= \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

(9) Find eigenvalues, then a basis for each eigenspace.

$$\det(A-\lambda I) = \begin{vmatrix} -3-\lambda & 4 \\ -1 & 1-\lambda \end{vmatrix}$$

$$= (-3-\lambda)(1-\lambda)+4$$

$$= \lambda^2+2\lambda+1$$

$$= (\lambda+1)^2$$

Set $\det(A-\lambda I)=0$: $(\lambda+1)^2=0$

$\lambda=-1,-1$

E_{-1} : $[A+I \mid \vec{0}]$

$$\left[\begin{array}{cc|c} -2 & 4 & 0 \\ -1 & 2 & 0 \end{array}\right]$$

$\frac{R_1}{-2}$ $$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ -1 & 2 & 0 \end{array}\right]$$

R_2+R_1 $$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2=t$

One free variable

$\Rightarrow$ E_{-1} has one basis vector

$\Rightarrow$ Not enough basis vectors to build P

$\Rightarrow$ A is not diagonalizable.

(13) $|A-\lambda I| = \begin{vmatrix} 1-\lambda & 2 & 1 \\ -1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix}$

$= (1-\lambda)\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - 2\begin{vmatrix} -1 & 1 \\ 1 & -\lambda \end{vmatrix} + 1\begin{vmatrix} -1 & -\lambda \\ 1 & 1 \end{vmatrix}$

$= (1-\lambda)(\lambda^2-1) - 2(\lambda-1) + 1(-1+\lambda)$

$= (1-\lambda)(\lambda^2-1) + 2(1-\lambda) - 1(1-\lambda)$

Factor

$= (1-\lambda)[\lambda^2-1+2-1]$

$= (1-\lambda)\lambda^2$

Set $|A-\lambda I| = 0$: $(1-\lambda)\lambda^2 = 0$

$\lambda = 0, 1$

E_0 : $[A \mid \vec{0}]$

$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right]$

$\begin{array}{l} \\ R_2+R_1 \\ R_3-R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$

$\frac{R_2}{2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \rightarrow$

(13) Cont'd

$$\begin{array}{l} R_1 - 2R_2 \\ \\ R_3 + R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = t$

One free variable

$\Rightarrow$ E_0 has one basis vector

$\Rightarrow$ Geometric multiplicity of $\lambda = 0$ is 1 but algebraic multiplicity of $\lambda = 0$ is 2

$\Rightarrow$ Cannot diagonalize A

(15) A is upper triangular

$\Rightarrow \lambda = 2, -2$

E_2: $[A - 2I \mid \vec{0}]$

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{array}\right]$$

$R_3/-4$ and $R_4/-4$

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$R_1 - 4R_4$ then reorder rows

$$\left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = s$ $\quad x_2 = t$ $\quad x_3 = 0$ $\quad x_4 = 0$

$\longrightarrow$

15 Cont'd

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} t$$

Basis for $E_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

E_{-2} : $[A + 2I \mid \vec{0}]$

$$\left[\begin{array}{cccc|c} 4 & 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1/4$ and $R_2/4$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = s$

$x_4 = t$

$x_1 + x_4 = 0 \Rightarrow x_1 = -t$

$x_2 = 0$

$$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} t$$

Basis for $E_{-2} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

A is diagonalizable.

$$P = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

(17) Find P and D first.

$$|A-\lambda I| = \begin{vmatrix} -1-\lambda & 6 \\ 1 & -\lambda \end{vmatrix}$$

$$= (-1-\lambda)(-\lambda) - 6$$

$$= \lambda^2 + \lambda - 6$$

$$= (\lambda+3)(\lambda-2)$$

Set $|A-\lambda I| = 0$: $(\lambda+3)(\lambda-2) = 0$

$\lambda = -3, 2$

E_{-3} : $[A+3I \mid \vec{0}]$

$$\left[\begin{array}{cc|c} 2 & 6 & 0 \\ 1 & 3 & 0 \end{array}\right]$$

$\frac{R_1}{2}$ $\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 1 & 3 & 0 \end{array}\right]$

$R_2 - R_1$ $\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_2 = t$

$x_1 + 3x_2 = 0 \Rightarrow x_1 = -3t$

$\vec{x} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} t$ Basis for $E_{-3} = \left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\}$

$\rightarrow$

(17) Cont'd

E_2 : $[A-2I \mid \bar{0}]$

$$\left[\begin{array}{cc|c} -3 & 6 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

$\frac{R_1}{-3}$ $$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 1 & -2 & 0 \end{array}\right]$$

$R_2 - R_1$ $$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = t$

$x_1 - 2x_2 = 0 \Rightarrow x_1 = 2t$

$\bar{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} t$ Basis for $E_2 = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

$P = \begin{bmatrix} -3 & 2 \\ 1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$ $D^{10} = \begin{bmatrix} (-3)^{10} & 0 \\ 0 & 2^{10} \end{bmatrix} = \begin{bmatrix} 59049 & 0 \\ 0 & 1024 \end{bmatrix}$ $P^{-1} = \frac{1}{-5}\begin{bmatrix} 1 & -2 \\ -1 & -3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$

$P^{-1}AP = D$

$A = PDP^{-1}$

$A^{10} = PD^{10}P^{-1}$

$= \frac{1}{5}\begin{bmatrix} -3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 59049 & 0 \\ 0 & 1024 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix} -177147 & 2048 \\ 59049 & 1024 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix} 179195 & -348150 \\ -58025 & 121170 \end{bmatrix}$

$= \begin{bmatrix} 35839 & -69630 \\ -11605 & 24234 \end{bmatrix}$

(23) Find P and D first.

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 2 & -2-\lambda & 2 \\ 0 & 1 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} -2-\lambda & 2 \\ 1 & 1-\lambda \end{vmatrix} - 1\begin{vmatrix} 2 & 2 \\ 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[(-2-\lambda)(1-\lambda) - 2\right] - \left[2(1-\lambda)\right]$$

$$= (1-\lambda)(\lambda^2 + \lambda - 4) - 2(1-\lambda)$$

$$= (1-\lambda)\left[\lambda^2 + \lambda - 4 - 2\right]$$

$$= (1-\lambda)(\lambda^2 + \lambda - 6)$$

$$= (1-\lambda)(\lambda+3)(\lambda-2)$$

Set $|A - \lambda I| = 0$: $(1-\lambda)(\lambda+3)(\lambda-2) = 0$

$\lambda = -3, 1, 2$

E_{-3} : $[A + 3I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 4 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right]$$

$\frac{R_1}{4}$ $$\left[\begin{array}{ccc|c} 1 & \frac{1}{4} & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right]$$

$R_2 - 2R_1$ $$\left[\begin{array}{ccc|c} 1 & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 2 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right]$$

$\rightarrow$

(23) Cont'd

$$R_2 \leftrightarrow R_3 \quad \left[\begin{array}{ccc|c} 1 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & \frac{1}{2} & 2 & 0 \end{array}\right]$$

$$\begin{array}{l} R_1 - \frac{1}{4}R_2 \\ \\ R_3 - \frac{1}{2}R_2 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = t$

$x_1 - x_3 = 0 \Rightarrow x_1 = t$

$x_2 + 4x_3 = 0 \Rightarrow x_2 = -4t$

$\vec{x} = \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} t$ Basis for $E_{-3} = \left\{ \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \right\}$

E_1 : $[A - I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 2 & -3 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$\frac{R_2}{2} \quad \left[\begin{array}{ccc|c} 1 & -\frac{3}{2} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

then $R_1 \leftrightarrow R_2$

$$\begin{array}{l} R_1 + \frac{3}{2}R_2 \\ \\ R_3 - R_2 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = t$

$x_1 + x_3 = 0 \Rightarrow x_1 = -t$

$x_2 = 0$

$\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} t$ Basis for $E_1 = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(23) Cont'd E_2:

$$[A - 2I \mid \vec{0}]$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 2 & -4 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\frac{R_1}{-1} \quad \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 2 & -4 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$R_2 - 2R_1 \quad \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\frac{R_2}{-2} \quad \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\begin{array}{l} R_1 + R_2 \\ \\ R_3 - R_2 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\uparrow$$

$$x_3 = t$$

$$x_1 - x_3 = 0 \Rightarrow x_1 = t$$

$$x_2 - x_3 = 0 \Rightarrow x_2 = t$$

$$\bar{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t \qquad \text{Basis for } E_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$P = \begin{bmatrix} 1 & -1 & 1 \\ -4 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$D^k = \begin{bmatrix} (-3)^k & 0 & 0 \\ 0 & 1^k & 0 \\ 0 & 0 & 2^k \end{bmatrix}$$

$$= \begin{bmatrix} (-3)^k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^k \end{bmatrix}$$

(23) Cont'd

Find P^{-1}:

$$[\,P \mid I\,]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} \\ R_2+4R_1 \\ R_3-R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -4 & 5 & 4 & 1 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} \\ \frac{R_2}{-4} \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{5}{4} & -1 & -\frac{1}{4} & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} R_1+R_2 \\ \\ R_3-2R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 \\ 0 & 1 & -\frac{5}{4} & -1 & -\frac{1}{4} & 0 \\ 0 & 0 & \frac{5}{2} & 1 & \frac{1}{2} & 1 \end{array}\right]$$

$$\begin{array}{c} \\ \\ \frac{2}{5}R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & 0 \\ 0 & 1 & -\frac{5}{4} & -1 & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \end{array}\right]$$

$$\begin{array}{c} R_1+\frac{1}{4}R_3 \\ R_2+\frac{5}{4}R_3 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} \frac{1}{10} & -\frac{1}{5} & \frac{1}{10} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \end{matrix}}_{P^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

$$P^{-1} = \frac{1}{10}\begin{bmatrix} 1 & -2 & 1 \\ -5 & 0 & 5 \\ 4 & 2 & 4 \end{bmatrix}$$

$$P^{-1}AP = D$$

$$A = PDP^{-1}$$

$$A^k = PD^kP^{-1}$$

$\longrightarrow$

(23) Cont'd

$$A^k = \frac{1}{10}\begin{bmatrix} 1 & -1 & 1 \\ -4 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} (-3)^k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^k \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ -5 & 0 & 5 \\ 4 & 2 & 4 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} (-3)^k & -1 & 2^k \\ -4(-3)^k & 0 & 2^k \\ (-3)^k & 1 & 2^k \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ -5 & 0 & 5 \\ 4 & 2 & 4 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} (-3)^k + 5 + 4\cdot 2^k & -2(-3)^k + 2\cdot 2^k & (-3)^k - 5 + 4\cdot 2^k \\ -4(-3)^k + 4\cdot 2^k & 8(-3)^k + 2\cdot 2^k & -4(-3)^k + 4\cdot 2^k \\ (-3)^k - 5 + 4\cdot 2^k & -2(-3)^k + 2\cdot 2^k & (-3)^k + 5 + 4\cdot 2^k \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5 + 2^{k+2} + (-3)^k}{10} & \frac{2^k - (-3)^k}{5} & \frac{-5 + 2^{k+2} + (-3)^k}{10} \\ \frac{2^{k+1} - 2(-3)^k}{5} & \frac{2^k + 4(-3)^k}{5} & \frac{2^{k+1} - 2(-3)^k}{5} \\ \frac{-5 + 2^{k+2} + (-3)^k}{10} & \frac{2^k - (-3)^k}{5} & \frac{5 + 2^{k+2} + (-3)^k}{10} \end{bmatrix}$$