Section 4.2

(1)

$$\begin{vmatrix} 1 & 0 & 3 \\ 5 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= 1(1) + 3(5)$$

$$= 16$$

$$\begin{vmatrix} 1 & 0 & 3 \\ 5 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - 5\begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix}$$

$$= 1(1) - 5(-3)$$

$$= 16$$

(3)

$$\begin{vmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix}$$

$$= 1(-1) + 1(1)$$

$$= 0$$

$$\begin{vmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} -1 & 0 \\ 1 & -1 \end{vmatrix}$$

$$= 1(-1) + 1(1)$$

$$= 0$$

(11)

$$\begin{vmatrix} a & b & 0 \\ 0 & a & b \\ a & 0 & b \end{vmatrix} = a\begin{vmatrix} a & b \\ 0 & b \end{vmatrix} - b\begin{vmatrix} 0 & b \\ a & b \end{vmatrix}$$

$$= a(ab) - b(-ab)$$

$$= a^2b + ab^2$$

(23)

$$\begin{vmatrix} -4 & 1 & 3 \\ 2 & -2 & 4 \\ 1 & -1 & 0 \end{vmatrix}$$

$R_1 \Leftrightarrow R_3$

$$= -\begin{vmatrix} 1 & -1 & 0 \\ 2 & -2 & 4 \\ -4 & 1 & 3 \end{vmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 + 4R_1$

$$= -\begin{vmatrix} 1 & -1 & 0 \\ 0 & 0 & 4 \\ 0 & -3 & 3 \end{vmatrix}$$

$R_2 \leftrightarrow R_3$

$$= \begin{vmatrix} 1 & -1 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & 4 \end{vmatrix}$$ upper triangular

$$= 1(-3)(4)$$

$$= -12$$

(27) $\begin{vmatrix} 3 & 1 & 0 \\ 0 & -2 & 5 \\ 0 & 0 & 4 \end{vmatrix}$ upper triangular

$= 3(-2)(4)$

$= -24$

(33) $\begin{vmatrix} 0 & 2 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{vmatrix}$

$R_1 \leftrightarrow R_2$
and $R_3 \leftrightarrow R_4$

$= (-1)(-1) \begin{vmatrix} -3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix}$ upper triangular

$= (-3)(2)(1)(4)$

$= -24$

(35) $\begin{vmatrix} 2a & 2b & 2c \\ d & e & f \\ g & h & i \end{vmatrix}$

$= 2 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$= 2(4)$

$= 8$

(37) $\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$

$= - \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$= -4$

(45)

$$\begin{vmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{vmatrix}$$

$$= k \begin{vmatrix} k+1 & 1 \\ -8 & k-1 \end{vmatrix} + k \begin{vmatrix} -k & 3 \\ k+1 & 1 \end{vmatrix}$$

$$= k\left[(k+1)(k-1)+8 \right] + k\left[-k-3(k+1) \right]$$

$$= k\left[k^2+7 \right] + k\left[-k-3k-3 \right]$$

$$= k\left[k^2+7 \right] + k\left[-4k-3 \right]$$

$$= k\left[k^2+7-4k-3 \right]$$

$$= k\left[k^2-4k+4 \right]$$

$$= k(k-2)^2$$

A is invertible if and only if $|A| \neq 0$

if and only if $k \neq 0, 2$

(49) $\det(B^{-1}A)$

$= \det(B^{-1}) \det(A)$

$= \frac{1}{\det B} \cdot \det A$

$= \frac{3}{-2}$

$= \frac{-3}{2}$

(51) $\det(3B^T)$

$= 3^n \det(B^T)$

$= 3^n \det(B)$

$= 3^n(-2)$

$= (-2)3^n$

(55) $A^2 = A$

$\det(A^2) = \det A$

$\det A \cdot \det A = \det A$

Let $\det A = x$

$x^2 = x$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0, 1$

$\det A = 0, 1$

57

$$|A| = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

$$|A_1| = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3$$

$$|A_2| = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$$

$$x = \frac{|A_1|}{|A|} = \frac{3}{2}$$

$$y = \frac{|A_2|}{|A|} = \frac{1}{-2} = \frac{-1}{2}$$

59

$$|A| = \begin{vmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 2$$

upper triangular

$$|A_1| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= 1(-2) + 1(0)$$

$$= -2$$

$$|A_2| = \begin{vmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

$\rightarrow$

$$|A_3| = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 2$$

upper triangular

$$x = \frac{|A_1|}{|A|} = \frac{-2}{2} = -1$$

$$y = \frac{|A_2|}{|A|} = \frac{0}{2} = 0$$

$$z = \frac{|A_3|}{|A|} = \frac{2}{2} = 1$$

(63) $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

cofactor matrix $C = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -2 & -2 & 2 \end{bmatrix}$

$$\operatorname{adj}(A) = C^T = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

$|A| = 2$ because A is upper triangular

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A)$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{bmatrix} \text{ or } \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$