(1) Show that $A\bar{v} = \lambda \bar{v}$

$$A\bar{v} = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$= 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The eigenvalue is $\lambda = 3$.

(3) Show that $A\bar{v} = \lambda \bar{v}$

$$A\bar{v} = \begin{bmatrix} -1 & 1 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 6 \end{bmatrix}$$

$$= -3 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$= -3\bar{v}$$

The eigenvalue is $\lambda = -3$.

(5) Show that $A\bar{v} = \lambda \bar{v}$

$$A\bar{v} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix}$$

$$= 3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$= 3\bar{v}$$

The eigenvalue is $\lambda = 3$.

(7) To find eigenvectors, solve $[A - \lambda I \mid \vec{0}]$

$$[A - 3I \mid \vec{0}]$$

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array}\right]$$

$$\frac{R_1}{-1} \quad \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 2 & -4 & 0 \end{array}\right]$$

$$R_2 - 2R_1 \quad \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x_2 = t$$

$$x_1 - 2x_2 = 0 \Rightarrow x_1 = 2t$$

$$\bar{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} t$$

One eigenvector is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$$A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} \checkmark$$

(11) To find eigenvectors, solve $[A - \lambda I \mid \vec{0}]$

$$[A + I \mid \vec{0}]$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right]$$

$$\frac{R_1}{2} \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right]$$

$$\begin{array}{c} R_2 + R_1 \\ R_3 - 2R_1 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\frac{R_2}{2} \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\uparrow$$

$$x_3 = t$$

$$x_1 + x_3 = 0 \Rightarrow x_1 = -t$$

$$x_2 + x_3 = 0 \Rightarrow x_2 = -t$$

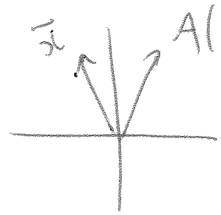
$$\vec{x} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} t$$

One eigenvector is $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

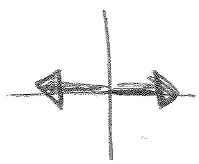
Another one is $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

$$A \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \checkmark$$

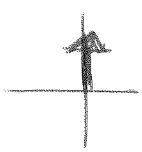
(13) Reflection in the y-axis



for which $\vec{x}$ are $\vec{x}$ and $A\vec{x}$ multiples of one another?



If $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ then $A\vec{x} = -1\vec{x}$

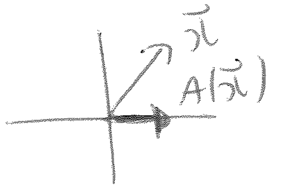


If $\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $A\vec{x} = 1\vec{x}$

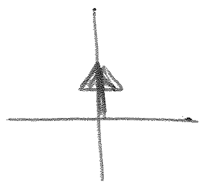
$\lambda = 1 \qquad E_1 = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

$\lambda = -1 \qquad E_{-1} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

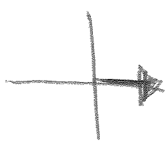
(15) Projection onto the x-axis



For which $\vec{x}$ are $\vec{x}$ and $A\vec{x}$ multiples of one another?



If $\vec{x} = \begin{bmatrix}0\\1\end{bmatrix}$ then $A\vec{x} = 0\,\vec{x}$

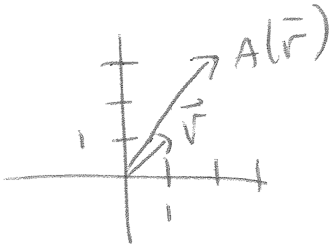


If $\vec{x} = \begin{bmatrix}1\\0\end{bmatrix}$ then $A\vec{x} = 1\,\vec{x}$

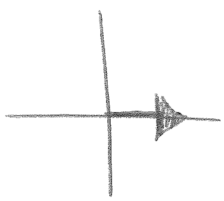
$\lambda = 1 \qquad E_1 = \text{span}\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$

$\lambda = 0 \qquad E_0 = \text{span}\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$

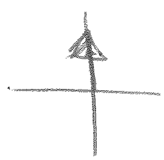
(17) Stretch by a factor of 2 horizontally and a factor of 3 vertically.



For which $\bar{x}$ are $\bar{x}$ and $A\bar{x}$ multiples of one another?



If $\bar{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ then $A\bar{x} = 2\bar{x}$



If $\bar{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $A\bar{x} = 3\bar{x}$

$\lambda = 2 \qquad E_2 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

$\lambda = 3 \qquad E_3 = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

(23) To find the eigenvalues solve

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(1-\lambda)+2=0$$

$$\lambda^2-5\lambda+4+2=0$$

$$\lambda^2-5\lambda+6=0$$

$$(\lambda-2)(\lambda-3)=0$$

$$\lambda=2,3$$

E_2: Solve $[A-2I \mid \vec{0}]$

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right]$$

$\frac{R_1}{2}$ $$\left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 2 & -1 & 0 \end{array}\right]$$

R_2-2R_1 $$\left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\uparrow$

$x_2=t$

$$x_1-\frac{1}{2}x_2=0 \implies x_1=\frac{1}{2}t$$

$$\bar{x}=\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}t \quad \text{or} \quad \bar{x}=\begin{bmatrix} 1 \\ 2 \end{bmatrix}t$$

Basis for $E_2 = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$

$\rightarrow$

(23) Cont'd

E_3: Solve $[A-3I \mid \vec{0}]$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \end{array}\right]$$

$$R_2 - 2R_1 \quad \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\uparrow$$

$x_2 = t$

$x_1 - x_2 = 0 \Rightarrow x_1 = t$

$$\bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t$$

Basis for $E_3 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

Illustration of $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

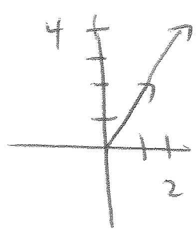
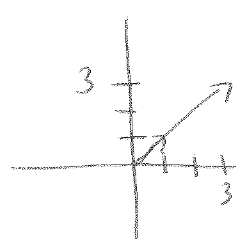


Illustration of $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$



(25) To find the eigenvalues solve

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & 5 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2 = 0$$

$$\lambda = 2, 2$$

E_2 : Solve $[A - 2I \mid \bar{0}]$

$$\left[\begin{array}{cc|c} 0 & 5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\frac{R_1}{5}$

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\uparrow$

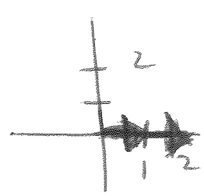
$x_1 = t$

$x_2 = 0$

$$\bar{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t$$

Basis for $E_2 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Illustration of $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$



(37) To find the eigenvalues solve

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} a-\lambda & b \\ 0 & d-\lambda \end{vmatrix}=0$$

$$(a-\lambda)(d-\lambda)=0$$

$$\lambda = a, d$$

We assume $a \neq d$.

E_a: Solve $[A-aI \mid \vec{0}]$

$$\left[\begin{array}{cc|c} 0 & b & 0 \\ 0 & d-a & 0 \end{array}\right]$$

$$\frac{R_1}{b} \quad \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & d-a & 0 \end{array}\right]$$

$$R_2-(d-a)R_1 \quad \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Note: If b happens to be 0 then do a row swap to get:

$$\left[\begin{array}{cc|c} 0 & d-a & 0 \\ 0 & b & 0 \end{array}\right]$$

The RREF is still $\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$x_1 = t$

$x_2 = 0$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t$$

Basis for $E_a = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Check: $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ✓

→

E_d : Solve $[A - dI \mid \vec{0}]$

$$\left[\begin{array}{cc|c} a-d & b & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\frac{R_1}{a-d} \quad \left[\begin{array}{cc|c} 1 & \frac{b}{a-d} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\uparrow$$
$$x_2 = t$$

$$x_1 + \frac{b}{a-d} x_2 = 0 \quad \Rightarrow \quad x_1 = \frac{-b}{a-d} t$$

$$\bar{x} = \begin{bmatrix} \frac{-b}{a-d} \\ 1 \end{bmatrix} t$$

$$\text{or} \quad \bar{x} = \begin{bmatrix} -b \\ a-d \end{bmatrix} t$$

$$\text{or} \quad \bar{x} = \begin{bmatrix} b \\ d-a \end{bmatrix} t$$

Basis for $E_d = \left\{ \begin{bmatrix} b \\ d-a \end{bmatrix} \right\}$

Check: $A \begin{bmatrix} b \\ d-a \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \begin{bmatrix} b \\ d-a \end{bmatrix} = \begin{bmatrix} ab + bd - ab \\ d(d-a) \end{bmatrix} = d \begin{bmatrix} b \\ d-a \end{bmatrix}$ ✓