Section 3.5

(1) $S = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} \right\}$

$= \left\{ y \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$= \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$

S is a subspace of $\mathbb{R}^2$

(3) $S = \left\{ \begin{bmatrix} x \\ 2x \end{bmatrix} \right\}$

$= \left\{ x \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

$= \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$

S is a subspace of $\mathbb{R}^2$

(7) $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ such that } x - y + z = 1 \right\}$

$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ such that } z = 1 - x + y \right\}$

$= \left\{ \begin{bmatrix} x \\ y \\ 1 - x + y \end{bmatrix} \right\}$

$\neq \left\{ x \begin{bmatrix} \# \\ \# \\ \# \end{bmatrix} + y \begin{bmatrix} \# \\ \# \\ \# \end{bmatrix} \right\}$

S is <u>not</u> a subspace of $\mathbb{R}^3$

(11) a) Is $\vec{b}$ in col(A)?

Let $c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}0\\1\end{bmatrix} + c_3\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$

$$\begin{array}{cccc} c_1 & c_2 & c_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 1 & 1 & 1 & 2 \end{array}\right]$$

$$R_2 - R_1 \quad \left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -1 \end{array}\right] \quad \text{RREF}$$

system is consistent (solvable)

$\boxed{\text{Yes}}$

b) Is $\vec{w}$ in row(A)?

Let $c_1[1\ 0\ -1] + c_2[1\ 1\ 1] = [-1\ 1\ 1]$

$$\begin{array}{cc} c_1 & c_2 \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{array}\right]$$

$$R_3 + R_1 \quad \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{array}\right]$$

$$R_3 - 2R_2 \quad \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{array}\right] \quad \text{REF}$$

System has no solution.

$\boxed{\text{No}}$

(15) Is $\bar{r}$ in null(A)?

Is $A\bar{r} = \vec{0}$?

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\boxed{\text{No}}$

(17) $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

$R_2 - R_1$ $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ RREF

a) Basis for row(A)

= nonzero rows of REF/RREF

$= \{ [1 \ 0 \ -1], [0 \ 1 \ 2] \}$

b) Basis for Col(A)

= Columns 1 and 2 of A

$= \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

c) Basis for null(A)

Solve $A\bar{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \rightsquigarrow \begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$\rightarrow$

(17) Cont'd

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \end{array}$$

$$\uparrow$$
$$x_3 = t$$

$$x_1 - x_3 = 0 \Rightarrow x_1 = t$$
$$x_2 + 2x_3 = 0 \Rightarrow x_2 = -2t$$

$$\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} t$$

Basis for null $(A) = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$

(27) Find a basis for span $\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$

Let $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ Find a basis for Col(A).

$R_2 + R_1$ $\begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

$R_2 \leftrightarrow R_3$ $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

$R_3 + R_2$ $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ REF

Use columns 1 and 2 of A

Basis $= \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(24) Find a basis for span $([2\ -3\ 1], [1\ -1\ 0], [4\ -4\ 1])$

Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -1 & 0 \\ 4 & -4 & 1 \end{bmatrix}$ Find a basis for row(A)

$R_1 \leftrightarrow R_2 \quad \begin{bmatrix} 1 & -1 & 0 \\ 2 & -3 & 1 \\ 4 & -4 & 1 \end{bmatrix}$

$\begin{matrix} R_2 - 2R_1 \\ R_3 - 4R_1 \end{matrix} \quad \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$\frac{R_2}{-1} \quad \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ REF

Basis = nonzero rows of REF

$= \{[1\ -1\ 0], [0\ 1\ -1], [0\ 0\ 1]\}$

Alternatively continue to RREF:

$R_1 + R_2 \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{matrix} R_1 + R_3 \\ R_2 + R_3 \end{matrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ RREF

Basis = nonzero rows of RREF

$= \{[1\ 0\ 0], [0\ 1\ 0], [0\ 0\ 1]\}$

(31) Find a basis consisting of some of the original vectors:

$[2\ \ -3\ \ 1],\ [1\ \ -1\ \ 0],\ [4\ \ -4\ \ 1]$

Let $A = \begin{bmatrix} 2 & 1 & 4 \\ -3 & -1 & -4 \\ 1 & 0 & 1 \end{bmatrix}$ Find a basis for Col(A).

$$R_1 \leftrightarrow R_3 \quad \begin{bmatrix} 1 & 0 & 1 \\ -3 & -1 & -4 \\ 2 & 1 & 4 \end{bmatrix}$$

$$\begin{matrix} R_2+3R_1 \\ R_3-2R_1 \end{matrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\frac{R_2}{-1} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$R_3-R_2 \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{REF}$$

Basis for Col (A)

= Columns 1, 2, 3 of A

$$= \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -4 \\ 1 \end{bmatrix} \right\}$$

(35) $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

$R_2 - R_1$ $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ REF

rank = # pivots in REF/RREF
= 2

nullity = # columns without pivots
= 1

(39) Recall that rank(A)
= #pivots in REF/RREF
= dim(row space of A)
= dim(column space of A)

A is 3×5 $\Rightarrow$ # pivots in REF/RREF is ≤ 3
$\Rightarrow$ dim(column space of A) ≤ 3
$\Rightarrow$ columns of A are linearly dependent

(41) If A is 3×5 then rank(A) could be 0, 1, 2, 3
nullity(A) = # columns − rank(A)
= 5 − rank(A)

nullity(A) could be 5, 4, 3, 2

(45) $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$ form a basis for $\mathbb{R}^3$

if and only if $\operatorname{rank}\left(\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}\right) = 3$

$$\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}$$

$$R_2 - R_1 \quad \begin{bmatrix}1&1&0\\0&-1&1\\0&1&1\end{bmatrix}$$

$$\frac{R_2}{-1} \quad \begin{bmatrix}1&1&0\\0&1&-1\\0&1&1\end{bmatrix}$$

$$R_3 - R_2 \quad \begin{bmatrix}1&1&0\\0&1&-1\\0&0&2\end{bmatrix} \text{ REF}$$

rank $= 3$

Yes, $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^3$.

51 Let $\vec{w} = c_1 \vec{b_1} + c_2 \vec{b_2}$

$$\begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 0 & 6 \\ 0 & -1 & 2 \end{array}\right] \end{array}$$

$$R_2 - 2R_1 \quad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 4 \\ 0 & -1 & 2 \end{array}\right]$$

$$\frac{R_2}{-2} \quad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{array}\right]$$

$$\begin{array}{l} R_1 - R_2 \\ \\ R_3 + R_2 \end{array} \quad \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

$c_1 = 3$
$c_2 = -2$

$\vec{w} = 3\vec{b}_1 - 2\vec{b_2}$

$[\vec{w}]_B = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ ← The coordinate vector contains the coefficients.