Section 3.2

(1) $X - 2A + 3B = 0$

$$X = 2A - 3B$$

$$= 2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 3\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ -3 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 \\ 3 & 5 \end{bmatrix}$$

(3) $2(A + 2B) = 3X$

$$2A + 4B = 3X$$

$$X = \frac{2}{3}A + \frac{4}{3}B$$

$$= \frac{2}{3}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \frac{4}{3}\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\left(2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 4\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \right)$$

$$= \frac{1}{3}\begin{bmatrix} -2 & 4 \\ 10 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} -2/3 & 4/3 \\ 10/3 & 4 \end{bmatrix}$$

(5) Let $c_1 A_1 + c_2 A_2 = B$

$$c_1\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 0 & 3 \end{bmatrix}$$

$\rightarrow$

$$\begin{array}{cc} c_1 & c_2 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 2 & 1 & 5 \\ -1 & 2 & 0 \\ 1 & 1 & 3 \end{array}\right]$$

$$\begin{array}{c} \\ R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 - R_1 \end{array} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

$$\begin{array}{c} \\ \\ R_3 - 2R_2 \\ R_4 - R_2 \end{array} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$c_1 = 2$$

$$c_2 = 1$$

$$2A_1 + A_2 = B$$

(7) Let $c_1 A_1 + c_2 A_2 + c_3 A_3 = B$

$$c_1 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{array}{ccc} c_1 & c_2 & c_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 2 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ \\ R_3 + R_1 \\ \\ R_5 - R_1 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 2 & 1 & 1 \\ 0 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rightarrow$

$\frac{R_3}{(-1)}$ then rearrange bottom 5 rows

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} \\ \\ R_3-2R_2 \\ R_4-2R_2 \\ \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 5 & 9 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\frac{R_4}{3}$ then $R_3 \leftrightarrow R_4$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 5 & 9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_4-5R_3 \quad \left[\begin{array}{ccc|c} & & & \\ & & & \\ 0 & 0 & 0 & -1 \\ & & & \end{array}\right]$$

The system has no solution.

Not possible to write B as a linear combination of A_1, A_2, A_3.

(9) Let $c_1\begin{bmatrix}1&2\\-1&1\end{bmatrix} + c_2\begin{bmatrix}0&1\\2&1\end{bmatrix} = \begin{bmatrix}w&x\\y&z\end{bmatrix}$

$$\begin{array}{ccc} c_1 & c_2 & \end{array}$$
$$\left[\begin{array}{cc|c}1&0&w\\2&1&x\\-1&2&y\\1&1&z\end{array}\right]$$

$$\begin{array}{r} \\ R_2-2R_1 \\ R_3+R_1 \\ R_4-R_1 \end{array} \left[\begin{array}{cc|c}1&0&w\\0&1&x-2w\\0&2&y+w\\0&1&z-w\end{array}\right]$$

$$\begin{array}{r} \\ \\ R_3-2R_2 \\ R_4-R_2 \end{array} \left[\begin{array}{cc|c}1&0&w\\0&1&x-2w\\0&0&\\0&0&\end{array}\right]$$

$\leftarrow y+w-2(x-2w)$
$= 5w-2x+y$

$z-w-(x-2w)$
$= w-x+z$

Each zero row leads to a condition:

$5w-2x+y=0 \implies y = 2x-5w$

$w-x+z=0 \implies z = x-w$

The span is $\left\{ \begin{bmatrix}w & x\\ 2x-5w & x-w\end{bmatrix} \right\}$

(11) Let $c_1\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}+c_2\begin{bmatrix}-1 & 2 & 0\\ 0 & 1 & 0\end{bmatrix}+c_3\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}a & b & c\\ d & e & f\end{bmatrix}$

$$\begin{array}{cccc} c_1 & c_2 & c_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & a\\ 0 & 2 & 1 & b\\ -1 & 0 & 1 & c\\ 0 & 0 & 0 & d\\ 1 & 1 & 0 & e\\ 0 & 0 & 0 & f \end{array}\right]$$

$$\begin{array}{l} \\ \\ R_3+R_1\\ \\ R_5-R_1\\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & -1 & 1 & a\\ 0 & 2 & 1 & b\\ 0 & -1 & 2 & c+a\\ 0 & 0 & 0 & d\\ 0 & 2 & -1 & e-a\\ 0 & 0 & 0 & f \end{array}\right]$$

$\frac{R_3}{-1}$ and rearrange bottom 5 rows

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & a\\ 0 & 1 & -2 & -a-c\\ 0 & 2 & 1 & b\\ 0 & 2 & -1 & e-a\\ 0 & 0 & 0 & d\\ 0 & 0 & 0 & f \end{array}\right]$$

$$\begin{array}{l} \\ \\ R_3-2R_2\\ R_4-2R_2\\ \\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & -1 & 1 & a\\ 0 & 1 & -2 & -a-c\\ 0 & 0 & 5 & b+2a+2c\\ 0 & 0 & 3 & e+a+2c\\ 0 & 0 & 0 & d\\ 0 & 0 & 0 & f \end{array}\right]\begin{array}{l} \\ \\ \leftarrow b-2(-a-c)\\ \leftarrow e-a-2(-a-c)\\ \\ \\ \end{array}$$

$$\begin{array}{l} \\ \\ \\ R_4-\frac{3}{5}R_3\\ \\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & -1 & 1 & a\\ 0 & 1 & -2 & -a-c\\ 0 & 0 & 5 & b+2a+2c\\ 0 & 0 & 0 & e-\frac{a}{5}-\frac{3b}{5}+\frac{4c}{5}\\ 0 & 0 & 0 & d\\ 0 & 0 & 0 & f \end{array}\right]$$

REF

$\leftarrow e+a+2c-\frac{3}{5}(b+2a+2c)$
$= e+a+2c-\frac{3}{5}b-\frac{6}{5}a-\frac{6}{5}c$
$= e-\frac{1}{5}a-\frac{3}{5}b+\frac{4}{5}c$

Each zero row of REF leads to a condition →

$$\boxed{\begin{array}{l} e - \frac{1}{5}a - \frac{3}{5}b + \frac{4}{5}c = 0 \\ -\frac{3}{5}b + \frac{4}{5}c + e = \frac{1}{5}a \\ -3b + 4c + 5e = a \end{array}}$$

$$\boxed{d=0}$$

$$\boxed{f=0}$$

The span is $\left\{ \begin{bmatrix} -3b+4c+5e & b & c \\ 0 & e & 0 \end{bmatrix} \right\}$

(13) Let $c_1 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + c_2 \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{array}{cc} c_1 & c_2 \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 4 & 0 \\ 2 & 3 & 0 \\ 3 & 2 & 0 \\ 4 & 1 & 0 \end{array}\right]$$

$$\begin{array}{l} \\ R_2-2R_1 \\ R_3-3R_1 \\ R_4-4R_1 \end{array} \left[\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & -5 & 0 \\ 0 & -10 & 0 \\ 0 & -15 & 0 \end{array}\right]$$

$$\frac{R_2}{-5} \left[\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & -10 & 0 \\ 0 & -15 & 0 \end{array}\right]$$

$$\begin{array}{l} \\ \\ R_3+10R_2 \\ R_4+15R_2 \end{array} \left[\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The only Solution is $c_1 = c_2 = 0$

The matrices are linearly independent.

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Let $AB = BA$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix} = \begin{bmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{bmatrix}$$

The conditions are:

$$a+2c = a+3b$$
$$b+2d = 2a+4b$$
$$3a+4c = c+3d$$
$$3b+4d = 2c+4d$$

No need to simplify these conditions.

35a) A and B are symmetric

$\Rightarrow A^T = A$ and $B^T = B$.

We need to show that $(A+B)^T = A+B$

$$(A+B)^T = A^T + B^T$$
$$= A + B$$

(37) A matrix M is skew-symmetric if $M^T = -M$.

(b) and (c) are skew-symmetric

(a) and (d) are <u>not</u> skew-symmetric

(42) We need to show that

$$(A - A^T)^T = -(A - A^T)$$

$$(A - A^T)^T = A^T - (A^T)^T$$
$$= A^T - A$$
$$= -(A - A^T) \checkmark$$