

① Not REF

③ REF and RREF

⑦ Not REF

$$\textcircled{9} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3 \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{This is REF}$$

$$R_1 - R_2 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 - R_3 \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{This is RREF}$$

$$\textcircled{11} \quad \begin{bmatrix} 3 & 5 \\ 5 & -2 \\ 2 & 4 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3 \quad \begin{bmatrix} 2 & 4 \\ 5 & -2 \\ 3 & 5 \end{bmatrix}$$

$$\frac{R_1}{2} \quad \begin{bmatrix} 1 & 2 \\ 5 & -2 \\ 3 & 5 \end{bmatrix}$$

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$$\begin{array}{l} R_2 - 5R_1 \\ R_3 - 3R_1 \end{array} \begin{bmatrix} 1 & 2 \\ 0 & -12 \\ 0 & -1 \end{bmatrix}$$

$$\frac{R_2}{(-12)} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$R_3 + R_2 \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{This is REF}$$

$$R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{This is RREF}$$

$$(13) \begin{bmatrix} 3 & -2 & -1 \\ 2 & -1 & -1 \\ 4 & -3 & -1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2 \begin{bmatrix} 2 & -1 & -1 \\ 3 & -2 & -1 \\ 4 & -3 & -1 \end{bmatrix}$$

$$\frac{R_1}{2} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 3 & -2 & -1 \\ 4 & -3 & -1 \end{bmatrix}$$

$$\begin{array}{l} R_2 - 3R_1 \\ R_3 - 4R_1 \end{array} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -1 & 1 \end{bmatrix}$$

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$$\begin{array}{l} R_2 \\ (-\frac{1}{2}) \end{array} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$R_3 + R_2 \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{This is REF}$$

$$R_1 + \frac{1}{2}R_2 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{This is RREF}$$

(23) Rank = # leading nonzero entries in the REF or RREF

Question 1 REF = $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ rank = 3

Question 2 in REF rank = 2

Question 3 in RREF rank = 2

Question 4 in RREF rank = 0

Question 5 RREF = $\begin{bmatrix} 1 & 0 & 3 & -4 & 0 \\ 0 & 1 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ rank = 2

Question 6 RREF = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ rank = 3

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(23) Cont'd

Question 7

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 - R_1 \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 + 2R_2 \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \leftrightarrow R_4 \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_4 + R_3 \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{in REF}$$

rank = 3

Question 8 in REF rank = 3

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$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ 4 & -1 & 1 & 4 \end{array}$$

$$\begin{array}{l} R_2 - 2R_1 \\ R_3 - 4R_1 \end{array} \begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ \hline 0 & -5 & 7 & -18 \\ 0 & -9 & 13 & -32 \end{array}$$

$$\frac{R_2}{(-5)} \begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ \hline 0 & 1 & -7/5 & 18/5 \\ 0 & -9 & 13 & -32 \end{array}$$

$$\begin{array}{l} R_1 - 2R_2 \\ R_3 + 9R_2 \end{array} \begin{array}{ccc|c} 1 & 0 & -1/5 & 9/5 \\ \hline 0 & 1 & -7/5 & 18/5 \\ 0 & 0 & 2/5 & 2/5 \end{array}$$

$$\frac{R_3}{(2/5)} \begin{array}{ccc|c} 1 & 0 & -1/5 & 9/5 \\ \hline 0 & 1 & -7/5 & 18/5 \\ 0 & 0 & 1 & 1 \end{array}$$

$$\begin{array}{l} R_1 + \frac{1}{5}R_3 \\ R_2 + \frac{7}{5}R_3 \end{array} \begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ \hline 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

(27)

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline 1 & -3 & -2 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 4 & 6 & 0 \end{array}$$

$$\begin{array}{l} R_2 + R_1 \\ R_3 - 2R_1 \end{array} \begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ \hline 0 & -1 & -1 & 0 \\ 0 & 10 & 10 & 0 \end{array}$$

$$\frac{R_2}{(-1)} \begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 10 & 10 & 0 \end{array}$$

$$\begin{array}{l} R_1 + 3R_2 \\ R_3 - 10R_2 \end{array} \begin{array}{ccc|c} \textcircled{1} & 0 & 1 & 0 \\ \hline 0 & \textcircled{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}$$

↑
 $x_3 = t$

$$x_2 + x_3 = 0 \rightarrow x_2 = -x_3 \rightarrow x_2 = -t$$

$$x_1 + x_3 = 0 \rightarrow x_1 = -x_3 \Rightarrow x_1 = -t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Equivalently

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

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$$\begin{array}{cc|c} r & s & \\ \hline 2 & 1 & 3 \\ 4 & 1 & 7 \\ 2 & 5 & -1 \end{array}$$

$$\frac{R_1}{2} \begin{array}{cc|c} 1 & \frac{1}{2} & \frac{3}{2} \\ \hline 4 & 1 & 7 \\ 2 & 5 & -1 \end{array}$$

$$\begin{array}{l} R_2 - 4R_1 \\ R_3 - 2R_1 \end{array} \begin{array}{cc|c} 1 & \frac{1}{2} & \frac{3}{2} \\ \hline 0 & -1 & 1 \\ 0 & 4 & -4 \end{array}$$

$$\frac{R_2}{(-1)} \begin{array}{cc|c} 1 & \frac{1}{2} & \frac{3}{2} \\ \hline 0 & 1 & -1 \\ 0 & 4 & -4 \end{array}$$

$$\begin{array}{l} R_1 - \frac{1}{2}R_2 \\ R_3 - 4R_2 \end{array} \begin{array}{cc|c} 1 & 0 & 2 \\ \hline 0 & 1 & -1 \\ 0 & 0 & 0 \end{array} \leftarrow \text{no info}$$

$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

(33)

$$\begin{array}{cccc|c} w & x & y & z & \\ \hline 1 & 1 & 2 & 1 & 1 \\ 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 1 & 2 \end{array}$$

→

$$\begin{array}{l} R_2 - R_1 \\ R_4 - R_1 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 0 & -2 & -3 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -2 & 0 & 1 \end{array} \right]$$

$$R_2 \leftrightarrow R_3 \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & -2 & -3 & 0 & -1 \\ 0 & 0 & -2 & 0 & 1 \end{array} \right]$$

$$\begin{array}{l} R_1 - R_2 \\ R_3 + 2R_2 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & -3 \\ 0 & 0 & -2 & 0 & 1 \end{array} \right]$$

$$R_4 - 2R_3 \left[\begin{array}{cccc|c} & & & & \\ & & & & \\ & & & & \\ 0 & 0 & 0 & 0 & 7 \end{array} \right]$$

The system has no solution.

(43)

$$\begin{array}{c} x \quad y \quad z \\ \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & -2 \end{array} \right] \end{array}$$

$$\begin{array}{l} R_2 - R_1 \\ R_3 - kR_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & k-1 & 1-k & 0 \\ 0 & 1-k & 1-k^2 & -2-k \end{array} \right]$$

(43) Cont'd

$k=1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{array} \right]$$

No Solution

$k \neq 1$

$$\frac{R_2}{(k-1)} \left[\begin{array}{ccc|c} 1 & 1 & k & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1-k & 1-k^2 & -2k \end{array} \right]$$

$$\begin{array}{l} R_1 - R_2 \\ R_3 - (1-k)R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & k+1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -(k+2)(k-1) & -2-k \end{array} \right]$$

$$\begin{aligned} & 1-k^2 - (1-k)(-1) \\ &= 1-k^2 + 1-k \\ &= -k^2 - k + 2 \\ &= -(k^2 + k - 2) \\ &= -(k+2)(k-1) \end{aligned}$$

$k=-2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

∞ -many solutions

$k \neq -2$

$$\frac{R_3}{-(k+2)(k-1)} \left[\begin{array}{ccc|c} 1 & 0 & k+1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \# \end{array} \right]$$

where we don't need to solve for #

Unique Solution

$\left\{ \begin{array}{l} \text{No solution if } k=1 \\ \infty\text{-many solutions if } k=-2 \\ \text{Unique solution if } k \neq 1, -2 \end{array} \right.$

$$(45) \begin{array}{ccc|c} x & y & z & \\ \hline 3 & 2 & 1 & -1 \\ 2 & -1 & 4 & 5 \end{array}$$

$$\frac{R_1}{3} \begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ \hline 2 & -1 & 4 & 5 \end{array}$$

$$R_2 - 2R_1 \begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ \hline 0 & -\frac{7}{3} & \frac{10}{3} & \frac{17}{3} \end{array}$$

$$-\frac{3}{7} R_2 \begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ \hline 0 & 1 & -\frac{10}{7} & -\frac{17}{7} \end{array}$$

$$R_1 - \frac{2}{3} R_2 \begin{array}{ccc|c} \textcircled{1} & 0 & \frac{9}{7} & \frac{9}{7} \\ \hline 0 & \textcircled{1} & -\frac{10}{7} & -\frac{17}{7} \end{array}$$

↑
 $z = t$

$$y - \frac{10}{7}z = -\frac{17}{7} \rightarrow y = -\frac{17}{7} + \frac{10}{7}t$$

$$x + \frac{9}{7}z = \frac{9}{7} \rightarrow x = \frac{9}{7} - \frac{9}{7}t$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9/7 \\ -17/7 \\ 0 \end{bmatrix} + t \begin{bmatrix} -9/7 \\ 10/7 \\ 1 \end{bmatrix} \text{ is the best answer.}$$

Note: Letting $t=1$ we get the point

$$\text{Alternatively } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ -9/7 \\ 10/7 \end{bmatrix}$$

We can scale the direction vector: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 9 \\ -10 \\ -7 \end{bmatrix}$

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Solve $\vec{p} + s\vec{u} = \vec{q} + t\vec{v}$

$$\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - t \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$$s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{array}{cc|c} s & t & \\ \hline 1 & -2 & -4 \\ 0 & -3 & 0 \\ 1 & -1 & -1 \end{array}$$

$$R_3 - R_1 \quad \begin{array}{cc|c} 1 & -2 & -4 \\ 0 & -3 & 0 \\ 0 & 1 & 3 \end{array}$$

$$\frac{R_2}{(-3)} \quad \begin{array}{cc|c} 1 & -2 & -4 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{array}$$

$$R_3 - R_2 \quad \begin{array}{cc|c} 1 & -2 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{array}$$

The system has no solution.

The lines don't intersect.