Complex Numbers

(1) $z = (1+2i)(4-6i)^2$

$= (1+2i)(16-48i+36\underbrace{i^2}_{-36})$

$= (1+2i)(16-48i-36)$

$= (1+2i)(-20-48i)$

$= -20-48i-40i-96i^2$

$= -20-88i+96$

$= 76-88i$

(2) a) $z = 2+7i$

$\bar{z} = 2-7i$

$|z| = \sqrt{a^2+b^2}$

$= \sqrt{2^2+7^2}$

$= \sqrt{53}$

b) $z = -3-5i$

$\bar{z} = -3+5i$

$|z| = \sqrt{a^2+b^2}$

$= \sqrt{(-3)^2+(-5)^2}$

$= \sqrt{34}$

3 a) $\frac{1}{z} = \frac{1}{1-5i} \cdot \frac{1+5i}{1+5i}$

$= \frac{1+5i}{1+25}$ $\boxed{(a+bi)(a-bi) = a^2+b^2}$

$= \frac{1+5i}{26}$

$= \frac{1}{26} + \frac{5}{26} i$

b) $\frac{1}{z} = \frac{1}{1+i} \cdot \frac{1-i}{1-i}$

$= \frac{1-i}{1+1}$

$= \frac{1-i}{2}$

$= \frac{1}{2} - \frac{i}{2}$

(4) $$\frac{z_1}{z_2} = \frac{(2+i)}{(-7+5i)} \cdot \frac{(-7-5i)}{(-7-5i)}$$

$$= \frac{-14 - 10i - 7i + 5}{49 + 25}$$

$$= \frac{-9 - 17i}{74}$$

$$= \frac{-9}{74} - \frac{17}{74} i$$

(5) a) $$|z_1| = \sqrt{a^2 + b^2}$$

$$= \sqrt{4 + 12}$$

$$= \sqrt{16}$$

$$= 4$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Add π if real coefficient is negative

$$\theta = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right)$$

$$= \tan^{-1}\sqrt{3}$$

$$= \frac{\pi}{3}$$

$$z_1 = 4\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$$

$\rightarrow$

$$|z_2| = \sqrt{a^2+b^2}$$

$$= \sqrt{\frac{1}{3} + \frac{1}{9}}$$

$$= \sqrt{\frac{4}{9}}$$

$$= \frac{2}{3}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$\boxed{\text{Add } \pi \text{ if real coefficient is negative}}$

$$\theta = \tan^{-1}\left(\frac{\left(\frac{1}{3}\right)}{\left(\frac{\sqrt{3}}{3}\right)}\right)$$

$$= \tan^{-1}\frac{1}{\sqrt{3}}$$

$$= \frac{\pi}{6}$$

$$z_2 = \frac{2}{3}\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$$

b) When computing $z_1 z_2$ in polar form:
lengths multiply
and angles add

$$|z_1|\,|z_2| = 4\left(\frac{2}{3}\right) = \frac{8}{3}$$

$$\theta_1 + \theta_2 = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$$

$$z_1 z_2 = \frac{8}{3}\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$$

or $\frac{8}{3}i$ in rectangular form

(5) c) when computing $\frac{z_1}{z_2}$ in polar form:

lengths divide

angles subtract

$$\frac{|z_1|}{|z_2|} = \frac{4}{\left(\frac{2}{3}\right)} = 6$$

$$\theta_1 - \theta_2 = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

$$\frac{z_1}{z_2} = 6\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$$

(6) let $z = 1 + i$

$$|z| = \sqrt{a^2+b^2}$$
$$= \sqrt{1+1}$$
$$= \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Add π if real coefficient is negative

$$= \tan^{-1} 1$$
$$= \frac{\pi}{4}$$

$$z = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$$

$$z^8 = \sqrt{2}^8\left[\cos\left(\frac{8\pi}{4}\right) + i\sin\left(\frac{8\pi}{4}\right)\right]$$
$$= 16\left[1 + i(0)\right]$$
$$= 16$$

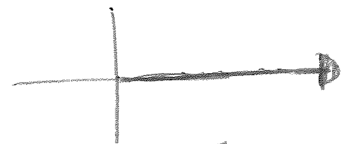
(7) Solve $z^6 - 64 = 0$

$$z^6 = 64$$

$$z = \sqrt[6]{64}$$

There will be 6 solutions.

Let $w = 64$



In polar form $w = 64[\cos 0 + i\sin 0]$

Want $w^{1/6} = 64^{1/6}\left[\cos \frac{0+2\pi n}{6} + i\sin \frac{0+2\pi n}{6}\right]$

$$= 2\left[\cos \frac{n\pi}{3} + i\sin \frac{n\pi}{3}\right]$$

for $n = 0, 1, 2, 3, 4, 5$

$$w_0^{1/6} = 2[\cos 0 + i\sin 0] = 2$$

$$w_1^{1/6} = 2\left[\cos \frac{\pi}{3} + i\sin \frac{\pi}{3}\right] = 2\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right] = 1 + \sqrt{3}\,i$$

$$w_2^{1/6} = 2\left[\cos \frac{2\pi}{3} + i\sin \frac{2\pi}{3}\right] = 2\left[\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right] = -1 + \sqrt{3}\,i$$

$$w_3^{1/6} = 2[\cos \pi + i\sin \pi] = -2$$

$$w_4^{1/6} = 2\left[\cos \frac{4\pi}{3} + i\sin \frac{4\pi}{3}\right] = 2\left[\frac{-1}{2} + i\left(\frac{-\sqrt{3}}{2}\right)\right] = -1 - \sqrt{3}\,i$$

$$w_5^{1/6} = 2\left[\cos \frac{5\pi}{3} + i\sin \frac{5\pi}{3}\right] = 2\left[\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right] = 1 - \sqrt{3}\,i$$

So $w^{1/6} = \pm 2,\ 1 \pm \sqrt{3}\,i,\ -1 \pm \sqrt{3}\,i$

or $z = \pm 2,\ 1 \pm \sqrt{3}\,i,\ -1 \pm \sqrt{3}\,i$

(8) $|wz| = |(a+bi)(c+di)|$

$= |ac + adi + bci + bdi^2|$

$= |(ac-bd) + (ad+bc)i|$

$= \sqrt{(ac-bd)^2 + (ad+bc)^2}$

$= \sqrt{a^2c^2 - 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2c^2}$

$= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2}$

$= \sqrt{a^2(c^2+d^2) + b^2(c^2+d^2)}$

$= \sqrt{(a^2+b^2)(c^2+d^2)}$

$= \sqrt{a^2+b^2}\sqrt{c^2+d^2}$

$= |w||z|$