1. [3 marks] $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for $\mathbb{R}^3$, where:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \text{ and } \mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Write $\mathbf{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ as a linear combination of the basis vectors.

Orthogonal basis

$$\Rightarrow \quad \vec{w} = \text{proj}_{\vec{v_1}} \vec{w} + \text{proj}_{\vec{v_2}} \vec{w} + \text{proj}_{\vec{v_3}} \vec{w}$$

$$= \frac{\vec{w} \cdot \vec{v_1}}{\|\vec{v_1}\|^2} \vec{v_1} + \dots$$

$$= \frac{x+2y+2z}{9} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \frac{2x+y-2z}{9} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} + \frac{2x-2y+z}{9} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

2. [7 marks] Find an orthogonal basis for span$\left(\begin{bmatrix}1\\0\\-1\\2\end{bmatrix}, \begin{bmatrix}2\\1\\2\\1\end{bmatrix}, \begin{bmatrix}-3\\5\\0\\1\end{bmatrix}\right)$.

Partial Basis $X = \left\{ \begin{bmatrix}1\\0\\-1\\2\end{bmatrix} \right\}$

$$\bar{v}_2 = \begin{bmatrix}2\\1\\2\\1\end{bmatrix} - \text{proj}_X \begin{bmatrix}2\\1\\2\\1\end{bmatrix}$$

$$= \begin{bmatrix}2\\1\\2\\1\end{bmatrix} - \frac{2}{6}\begin{bmatrix}1\\0\\-1\\2\end{bmatrix}$$

$$3\bar{v}_2 = 3\begin{bmatrix}2\\1\\2\\1\end{bmatrix} - \begin{bmatrix}1\\0\\-1\\2\end{bmatrix}$$

$$= \begin{bmatrix}5\\3\\7\\1\end{bmatrix}$$

Partial Basis $X = \left\{ \begin{bmatrix}1\\0\\-1\\2\end{bmatrix}, \begin{bmatrix}5\\3\\7\\1\end{bmatrix} \right\}$

$$\bar{v}_3 = \begin{bmatrix}-3\\5\\0\\2\end{bmatrix} - \text{proj}_X \begin{bmatrix}-3\\5\\0\\2\end{bmatrix}$$

$$= \begin{bmatrix}-3\\5\\0\\2\end{bmatrix} - \text{proj}_{\begin{bmatrix}1\\0\\-1\\2\end{bmatrix}} \begin{bmatrix}-3\\5\\0\\2\end{bmatrix} - \text{proj}_{\begin{bmatrix}5\\3\\7\\1\end{bmatrix}} \begin{bmatrix}-3\\5\\0\\2\end{bmatrix}$$

$$= \begin{bmatrix}-3\\5\\0\\2\end{bmatrix} - \frac{1}{6}\begin{bmatrix}1\\0\\-1\\2\end{bmatrix} - \frac{2}{84}\begin{bmatrix}5\\3\\7\\1\end{bmatrix}$$

$$42\bar{v}_3 = 42\begin{bmatrix}-3\\5\\0\\2\end{bmatrix} - 7\begin{bmatrix}1\\0\\-1\\2\end{bmatrix} - \begin{bmatrix}5\\3\\7\\1\end{bmatrix}$$

$$= \begin{bmatrix}-138\\207\\0\\69\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}-2\\3\\0\\1\end{bmatrix}$$

Orthogonal Basis $= \left\{ \begin{bmatrix}1\\0\\-1\\2\end{bmatrix}, \begin{bmatrix}5\\3\\7\\1\end{bmatrix}, \begin{bmatrix}-2\\3\\0\\1\end{bmatrix} \right\}$

3. [5 marks] Let $W = \text{span}\left(\begin{bmatrix}1\\0\\-8\\121\\-43\end{bmatrix}, \begin{bmatrix}0\\1\\9\\-140\\22\end{bmatrix}, \begin{bmatrix}3\\0\\-23\\347\\-126\end{bmatrix}\right)$.

Find a basis for $W^{\perp}$.

Solve
$$\left[\begin{array}{ccccc|c}1 & 0 & -8 & 121 & -43 & 0\\0 & 1 & 9 & -140 & 22 & 0\\3 & 0 & -23 & 347 & -126 & 0\end{array}\right]$$

$$R_3 - 3R_1 \quad \left[\begin{array}{ccccc|c}1 & 0 & -8 & 121 & -43 & 0\\0 & 1 & 9 & -140 & 22 & 0\\0 & 0 & 1 & -16 & 3 & 0\end{array}\right]$$

$$\begin{array}{l}R_1 + 8R_3\\R_2 - 9R_3\end{array} \quad \begin{array}{c}\begin{array}{ccccc}x_1 & x_2 & x_3 & x_4 & x_5\end{array}\\ \left[\begin{array}{ccccc|c}1 & 0 & 0 & -7 & -19 & 0\\0 & 1 & 0 & 4 & -5 & 0\\0 & 0 & 1 & -16 & 3 & 0\end{array}\right]\end{array}$$

$x_4 = s$ $\quad$ $x_5 = t$

$x_1 - 7x_4 - 19x_5 = 0 \Rightarrow x_1 = 7s + 19t$

$x_2 + 4x_4 - 5x_5 = 0 \Rightarrow x_2 = -4s + 5t$

$x_3 - 16x_4 + 3x_5 = 0 \Rightarrow x_3 = 16s - 3t$

$$\bar{x} = \begin{bmatrix}7\\-4\\16\\1\\0\end{bmatrix} s + \begin{bmatrix}19\\5\\-3\\0\\1\end{bmatrix} t$$

Basis for $W^{\perp} = \left\{ \begin{bmatrix}7\\-4\\16\\1\\0\end{bmatrix}, \begin{bmatrix}19\\5\\-3\\0\\1\end{bmatrix} \right\}$

4. [4 marks] $A = \begin{bmatrix} 5 & 2 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 5 \end{bmatrix}$ has characteristic equation $(\lambda - 4)^2(\lambda - 5) = 0$.

a) Find the algebraic multiplicity of $\lambda = 4$.

$\boxed{2}$

b) Find the geometric multiplicity of $\lambda = 4$.

$[A - 4I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$R_2 + R_1$
$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$R_1 - 2R_2$

$R_3 - R_2$
$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_3 = t$

(Geometric Multiplicity of $\lambda = 4$) = # of basis vectors for E_4
= # of parameters in solution to $[A - 4I \mid \vec{0}]$
= $\boxed{1}$

c) Is A diagonalizable? Explain.

No.

(Geometric Multiplicity of $\lambda = 4$) < (Algebraic Multiplicity of $\lambda = 4$)

5. [6 marks] The matrix A has eigenvalue 2 corresponding to the eigenvector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and eigenvalue 3 corresponding to the eigenvector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Find the top-left entry of A^n.

$$P = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \qquad P^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad D^n = \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix}$$

$$P^{-1}AP = D$$
$$AP = PD$$
$$A = PDP^{-1}$$
$$A^n = PD^nP^{-1}$$
$$= \frac{1}{5}\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$$
$$= \frac{1}{5}\begin{bmatrix} 3 \cdot 2^n & 2 \cdot 3^n \\ 2^{n+1} & 3^{n+1} \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$$
$$= \frac{1}{5}\begin{bmatrix} 9 \cdot 2^n - 4 \cdot 3^n & \\ & \end{bmatrix}$$

The top-left entry of A^n is $\frac{9(2^n) - 4(3^n)}{5}$