Complex Numbers (Appendix C in text)

Def

i is the imaginary number such that $i^2 = -1$

If a, b are real numbers then $z = a + bi$ is a complex number

Ex: $z_1 = -2 + 6i \qquad z_2 = 4 + 5i$

a) $-7z_1 = 14 - 42i$

b) $z_1 + z_2 = 2 + 11i$

c) $z_1 - z_2 = -6 + i$

d) $z_1 z_2 = (-2 + 6i)(4 + 5i)$

$= -8 - 10i + 24i - 30$

$= -38 + 14i$

def

The complex conjugate of $z = a+bi$ is $\bar{z} = a-bi$

Ex: Let $z = a+bi$. Show that $z\bar{z} = a^2+b^2$

$$z\bar{z} = (a+bi)(a-bi)$$
$$= a^2 - abi + abi - b^2 i^2$$
$$= a^2 + b^2$$

(a nonnegative real number)

Ex: Let $z_1 = 4+9i$ $\quad z_2 = -3+5i$

a) $\frac{1}{z_1} = \frac{1}{4+9i} \cdot \frac{4-9i}{4-9i}$

$$= \frac{4-9i}{4^2+9^2}$$
$$= \frac{4-9i}{97}$$
$$= \frac{4}{97} - \frac{9}{97}i$$

b) $\frac{z_1}{z_2} = \frac{4+9i}{-3+5i} \cdot \frac{-3-5i}{-3-5i}$

$$= \frac{-12-20i-27i+45}{34}$$
$$= \frac{33}{34} - \frac{47}{34}i$$

Def

The <u>length</u> or <u>modulus</u> of $z = a + bi$

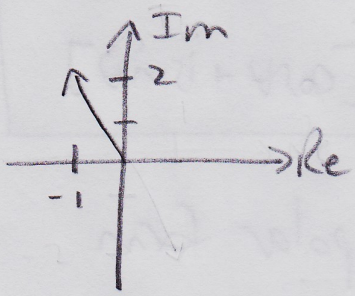
is $|z| = \sqrt{a^2 + b^2}$

The <u>principal argument</u> of $z = a + bi$

is the angle $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ $(+\pi?)$

We assume $-\pi < \theta \leq \pi$

<u>Ex</u>: $z = -1 + 2i$

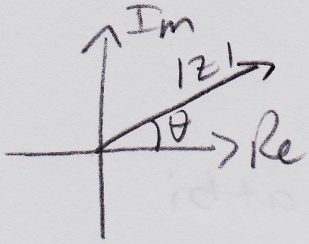


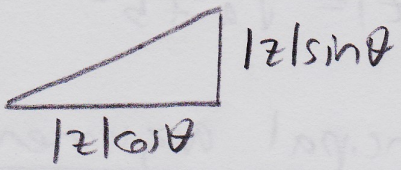
$|z| = \sqrt{5}$

$\theta = \tan^{-1}\left(\frac{2}{-1}\right) + \pi$

≈ 2.03 rads

Ex: Show that $z = |z|[\cos\theta + i\sin\theta]$





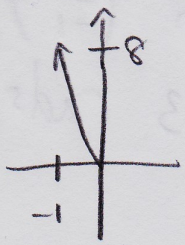
$$z = |z|\cos\theta + i|z|\sin\theta$$
$$= |z|[\cos\theta + i\sin\theta]$$

Def

Rectangular Form	$z = a + bi$
Polar form	$z = \lvert z\rvert[\cos\theta + i\sin\theta]$

Ex: Express $z = -1 + 8i$ in polar form

$$|z| = \sqrt{a^2 + b^2}$$
$$= \sqrt{1 + 64}$$
$$= \sqrt{65}$$



$$\theta = \tan^{-1}\left(\frac{b}{a}\right) \quad (+\pi\,?)$$
$$= \tan^{-1}\left(\frac{8}{-1}\right) + \pi$$
$$= \pi - \tan^{-1} 8$$

$$z = |z|[\cos\theta + i\sin\theta]$$
$$= \sqrt{65}\left[\cos\left(\pi - \tan^{-1} 8\right) + i\sin\left(\pi - \tan^{-1} 8\right)\right]$$

Multiplication and Division in Polar Form

FACT

Let $z_1 = |z_1|[\cos\theta_1 + i\sin\theta_1]$

$z_2 = |z_2|[\cos\theta_2 + i\sin\theta_2]$

$$z_1 z_2 = |z_1||z_2|[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$$

Ex: Find $\frac{z_1}{z_2}$ by converting to polar form

$z_1 = 9 + 3\sqrt{3}\,i \qquad z_2 = 4\sqrt{3} - 12i$

z_1: $|z_1| = \sqrt{81+27}$

$= \sqrt{108}$

$= 6\sqrt{3}$

$\theta_1 = \tan^{-1}\left(\frac{3\sqrt{3}}{9}\right)$

$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{\pi}{6}$

$$z_2 = \sqrt{16 \cdot 3 + 144}$$

$$= \sqrt{192}$$

$$= 8\sqrt{3}$$

$$\theta_2 = \tan^{-1}\left(\frac{-12}{4\sqrt{3}}\right)$$

$$= \tan^{-1}(-\sqrt{3})$$

$$= -\frac{\pi}{3}$$

$$\frac{|z_1|}{|z_2|} = \frac{6\sqrt{3}}{8\sqrt{3}} = \frac{3}{4}$$

$$\theta_1 - \theta_2 = \frac{\pi}{6} - \left(-\frac{\pi}{3}\right) = \frac{\pi}{2}$$

$$\frac{z_1}{z_2} = \frac{3}{4}\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$$

or $\frac{3}{4}i$

De Moivre's Formula

Let n be a positive integer

If $z = |z|[\cos\theta + i\sin\theta]$

then $z^n = |z|^n[\cos(n\theta) + i\sin(n\theta)]$

Ex: Find $(1-i)^{21}$

Let $z = 1-i$

$|z| = \sqrt{2}$ $\quad \theta = \tan^{-1}(-1) = \frac{-\pi}{4}$

$z = \sqrt{2}\left[\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right]$

$z^{21} = \sqrt{2}^{21}\left[\cos\frac{-21\pi}{4} + i\sin\frac{-21\pi}{4}\right]$

$\frac{-21\pi}{4} + 6\pi = \frac{3\pi}{4}$

$= \sqrt{2}^{20}\sqrt{2}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$

$= 1024\sqrt{2}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$ (polar)

or $1024\sqrt{2}\left[\frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right]$

$= -1024 + 1024i$ (rectangular)

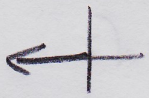
FACT

$z = |z|[\cos\theta + i\sin\theta]$ has n different n^{th} roots $(n = 2, 3, \dots)$

$$z^{1/n} = |z|^{1/n}\left[\cos\frac{\theta+2\pi\alpha}{n} + i\sin\frac{\theta+2\pi\alpha}{n}\right]$$

$\alpha = 0, 1, \dots, n-1$

Ex: Find all the cube roots of -1



$z = -1$

$z = 1[\cos\pi + i\sin\pi]$

$$z^{1/3} = 1^{1/3}\left[\cos\frac{\pi+2\pi\alpha}{3} + i\sin\frac{\pi+2\pi\alpha}{3}\right]$$

$\alpha = 0, 1, 2$

α	$\frac{\pi+2\pi\alpha}{3}$
0	$\frac{\pi}{3}$
1	$\frac{3\pi}{3} = \pi$
2	$\frac{5\pi}{3}$

$$z_1^{1/3} = 1\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right] = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$z_2^{1/3} = 1[\cos\pi + i\sin\pi] = -1$$

$$z_3^{1/3} = 1\left[\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right] = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Ex: Solve $x^2 - 2i = 0$

$x^2 = 2i$

$x = (2i)^{1/2}$ (two solutions)

$z = 2i$

$z = 2\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$

$$z^{1/2} = 2^{1/2}\left[\cos\frac{\frac{\pi}{2}+2\pi\alpha}{2} + \sin\frac{\frac{\pi}{2}+2\pi\alpha}{2}\right]$$

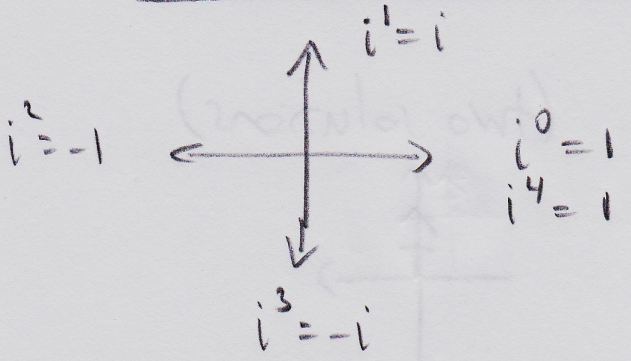
$\alpha = 0, 1$

α	$\frac{\frac{\pi}{2}+2\pi\alpha}{2}$
0	$\frac{\left(\frac{\pi}{2}\right)}{2} = \frac{\pi}{4}$
1	$\frac{\frac{\pi}{2}+2\pi}{2} = \frac{\pi+4\pi}{4} = \frac{5\pi}{4}$

$z_1^{1/2} = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] = 1+i$

$z_2^{1/2} = \sqrt{2}\left[\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right] = -1-i$

Powers of i



Ex: $i^{271} = i^{268} \cdot i^3 = i^3 = -i$

Ex: Show that $e^{i\theta} = \cos\theta + i\sin\theta$

given $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots$$

$$= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \dots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \dots$$

$$= 1 - \frac{\theta^2}{2!} + \dots + i\left[\theta - \frac{\theta^3}{3!} + \dots\right]$$

$$= \cos\theta + i\sin\theta$$

Three forms for a Complex Number

Rectangular	$z = a+bi$
Polar	$z = \|z\|[\cos\theta + i\sin\theta]$
Exponential	$z = \|z\|e^{i\theta}$

Complex Eigenvalues and Eigenvectors

Ex: $A = \begin{bmatrix} 3 & -13 \\ 5 & 1 \end{bmatrix}$

Find a basis for each eigenspace

λ:

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & -13 \\ 5 & 1-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(1-\lambda) + 65 = 0$$

$$\lambda^2 - 4\lambda + 68 = 0$$

$$\lambda = \frac{4 \pm \sqrt{-256}}{2}$$

$$\lambda = \frac{4 \pm 16i}{2}$$

$$\lambda = 2 \pm 8i$$

E_{2+8i} : $[A-(2+8i)I \mid \vec{0}]$

$$\left[\begin{array}{cc|c} 3-2-8i & -13 & 0 \\ 5 & 1-2-8i & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1-8i & -13 & 0 \\ 5 & -1-8i & 0 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\left[\begin{array}{cc|c} 5 & -1-8i & 0 \\ 1-8i & -13 & 0 \end{array}\right]$$

$R_1/5$

$$\left[\begin{array}{cc|c} 1 & \frac{-1-8i}{5} & 0 \\ 1-8i & -13 & 0 \end{array}\right]$$

$R_2-(1-8i)R_1$

$$\left[\begin{array}{cc|c} 1 & \frac{-1-8i}{5} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$-13-(1-8i)\frac{(-1-8i)}{5}$$
$$=-13+\frac{(-1+8i)(-1-8i)}{5}$$
$$=-13+\frac{65}{5}$$
$$=0$$

$x_2 = t$

$x_1+\left(\frac{-1-8i}{5}\right)x_2=0 \rightarrow x_1=\frac{1+8i}{5}t$

$$\vec{x}=\begin{bmatrix}\frac{1+8i}{5}\\ 1\end{bmatrix}t \text{ or } \begin{bmatrix}1+8i\\ 5\end{bmatrix}t$$

Basis $=\left\{\begin{bmatrix}1+8i\\ 5\end{bmatrix}\right\}$

E_{2-8i} : $[A-(2-8i)I \mid \bar{0}]$

$$\left[\begin{array}{cc|c} 3-2+8i & -13 & 0 \\ 5 & 1-2+8i & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1+8i & -13 & 0 \\ 5 & -1+8i & 0 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\left[\begin{array}{cc|c} 5 & -1+8i & 0 \\ 1+8i & -13 & 0 \end{array}\right]$$

$R_1/5$

$$\left[\begin{array}{cc|c} 1 & \frac{-1+8i}{5} & 0 \\ 1+8i & -13 & 0 \end{array}\right]$$

$R_2-(1+8i)R_1$

$$\left[\begin{array}{cc|c} 1 & \frac{-1+8i}{5} & \\ 0 & 0 & 0 \end{array}\right]$$

$$-13-(1+8i)\left(\frac{-1+8i}{5}\right)$$
$$=-13+\frac{(-1-8i)(-1+8i)}{5}$$
$$=-13+\frac{65}{5}$$
$$=0$$

$\boxed{x_2 = t}$

$x_1 + \frac{(-1+8i)}{5}x_2 = 0 \rightarrow \boxed{x_1 = \frac{1-8i}{5}t}$

$$\bar{x} = \begin{bmatrix} \frac{1-8i}{5} \\ 1 \end{bmatrix} t = \begin{bmatrix} 1-8i \\ 5 \end{bmatrix} t$$

$$\text{Basis} = \left\{ \begin{bmatrix} 1-8i \\ 5 \end{bmatrix} \right\}$$