251DX02Mar9

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3.6 Linear Transformations

Ex: Find $[T^{-1}]$, if it exists.

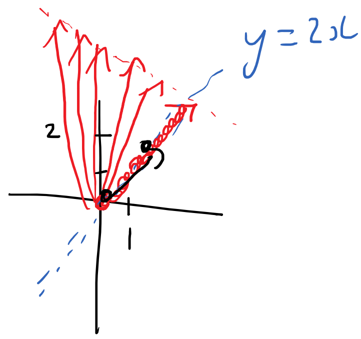
a) T: rotation by $-60°$

T^{-1}: rotation by $60°$

$$[T^{-1}] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \theta = 60°$$

$$= \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

b) T: projection onto $y = 2x$



Method I: T^{-1} does not exist

$[T^{-1}]$ does not exist

Method II:

$$[T] = \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix} \quad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$\det[T] = 0$

$$[T^{-1}] = [T]^{-1}$$

$[T^{-1}]$ does not exist

Ex: T is linear.

$$T\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}1\\7\end{bmatrix}$$

$$T\begin{bmatrix}4\\5\end{bmatrix} = \begin{bmatrix}2\\6\end{bmatrix}$$

Find $T\begin{bmatrix}28\\31\end{bmatrix}$.

1) $$\begin{bmatrix}28\\31\end{bmatrix} = c_1\begin{bmatrix}2\\3\end{bmatrix} + c_2\begin{bmatrix}4\\5\end{bmatrix}$$

$$\begin{array}{cc} c_1 & c_2 & \\ \end{array}$$
$$\left[\begin{array}{cc|c} 2 & 4 & 28\\ 3 & 5 & 31\end{array}\right]$$

$\vdots$

$$\begin{array}{cc} c_1 & c_2 & \\ \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 0 & -8\\ 0 & 1 & 11\end{array}\right]$$

$$\begin{bmatrix}28\\31\end{bmatrix} = -8\begin{bmatrix}2\\3\end{bmatrix} + 11\begin{bmatrix}4\\5\end{bmatrix}$$

2) $$T\left(\begin{bmatrix}28\\31\end{bmatrix}\right) = T\left(-8\begin{bmatrix}2\\3\end{bmatrix} + 11\begin{bmatrix}4\\5\end{bmatrix}\right)$$

$$\boxed{\begin{array}{l} T \text{ is linear} \Longrightarrow \\ T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v}) \\ T(c\vec{u}) = c\,T(\bar{u}) \\ \quad c: \text{real } \# \end{array}}$$

$$= T\left(-8\begin{bmatrix}2\\3\end{bmatrix}\right) + T\left(11\begin{bmatrix}4\\5\end{bmatrix}\right)$$

$$= -8\boxed{T\left(\begin{bmatrix}2\\3\end{bmatrix}\right)} + 11\boxed{T\left(\begin{bmatrix}4\\5\end{bmatrix}\right)}$$

$$= -8\begin{bmatrix}1\\ \cdot\end{bmatrix} + 11\begin{bmatrix}2\\ \cdot\end{bmatrix}$$

$$= -8\begin{bmatrix}1\\7\end{bmatrix} + 11\begin{bmatrix}2\\6\end{bmatrix}$$

$$= \begin{bmatrix}14\\10\end{bmatrix}$$

4.1 Eigenvectors and Eigenvalues

$$A\vec{x} = \lambda\vec{x} \qquad (\vec{x} \neq \vec{0})$$

eigenvector   eigenvalue

Ex: Find all the eigenvectors of $A = \begin{bmatrix}7 & 0 & -3\\-9 & -2 & 3\\18 & 0 & -8\end{bmatrix}$ corresponding to $\lambda = 1$.

Solve $[A - \lambda I \mid \vec{0}]$

$[A - I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c}6 & 0 & -3 & 0\\-9 & -3 & 3 & 0\\18 & 0 & -9 & 0\end{array}\right]$$

$$\frac{R_1}{6}\left[\begin{array}{ccc|c}1 & 0 & -\frac{1}{2} & 0\\-9 & -3 & 3 & 0\\18 & 0 & -9 & 0\end{array}\right]$$

$$18 \quad 0 \quad -9 \mid 0$$

$$\begin{array}{l} R_2 + 9R_1 \\ R_3 - 18R_1 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & -3 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \text{RREF}$$

$$x_3 = t$$
$$x_1 = \frac{t}{2}$$
$$x_2 = -\frac{t}{2}$$

$$\vec{x} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} t \qquad (\vec{x} \neq \vec{0})$$

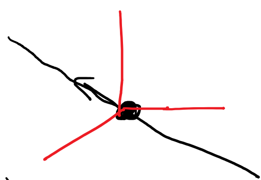
or

$$\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} t \qquad (\vec{x} \neq \vec{0})$$

$A\vec{0} = \lambda \vec{0}$ is trivial

Follow-Up #1: Find the eigenspace E_1 for A

$$E_1 = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right)$$



(eigenspaces includes $\vec{0}$ for convenience)

Follow-Up #2: Find a basis for the eigenspace E_1

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$$

Ex: Find all eigenvectors of $A = \begin{bmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{bmatrix}$

corresponding to $\lambda = -2$.

Solve $[A - \lambda I \mid \vec{0}]$

$[A + 2I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 9 & 0 & -3 & 0 \\ -9 & 0 & 3 & 0 \\ 18 & 0 & -6 & 0 \end{array}\right]$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{RREF}$$

$x_2 = s$ $\quad x_3 = t$

$x_1 = \frac{t}{3}$

$$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} \frac{1}{3} \\ 0 \\ 1 \end{bmatrix} t \qquad (\vec{x} \neq \vec{0})$$

or $$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} t \qquad (\vec{x} \neq \vec{0})$$

Check :

$$A\begin{bmatrix}1\\0\\3\end{bmatrix} = -2\begin{bmatrix}1\\0\\3\end{bmatrix} \checkmark$$

$$A\begin{bmatrix}0\\1\\0\end{bmatrix} = -2\begin{bmatrix}0\\1\\0\end{bmatrix} \checkmark$$

$$A\begin{bmatrix}1\\-1\\2\end{bmatrix} = 1\begin{bmatrix}1\\-1\\2\end{bmatrix} \checkmark$$

Can scale eigenvectors :

$$A\begin{bmatrix}0\\99\\0\end{bmatrix} = -2\begin{bmatrix}0\\99\\0\end{bmatrix} \checkmark$$