251DX02Mar8

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3.5 Subspaces / Basis

Ex: Find the coordinate vector of $\vec{v} = \begin{bmatrix} 12 \\ 6 \\ -4 \end{bmatrix}$

with respect to basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 9 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \right\}$

Let $\begin{bmatrix} 12 \\ 6 \\ -4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 9 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$ $\quad c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3$

$$\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & -2 & 4 & 12 \\ 1 & 1 & 1 & 6 \\ 9 & 3 & 1 & -4 \end{array}\right] \end{array}$$

$\vdots$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Coordinate vector $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$

Fundamental Theorem of Invertible Matrices

Let A be an $n \times n$ matrix.
Statements are all true or all false.

a) A is invertible

b) $A\vec{x} = \vec{b}$ has a unique solution.

d) RREF of A is I

e) A is a product of elementary matrices.

$\boxed{\text{j) Columns of } A \text{ are a basis for } \mathbb{R}^n}$

g) Columns of A form a basis for $\mathbb{R}^n$

h) $\det A \neq 0$

Ex: Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\vdots$$

$$\det A = 0$$

$\boxed{\text{No}}$

3.6 Linear Transformation

Ex: a) Is $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-2y \\ 3y \end{bmatrix}$ linear?

$\boxed{\text{T is linear if and only if T is a matrix transformation}}$

$$\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-2y \\ 3y \end{bmatrix}$$

YES

b) Is $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-2 \\ 3y \end{bmatrix}$ linear?

$$\begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-2 \\ 3y \end{bmatrix}$$

no variables allowed

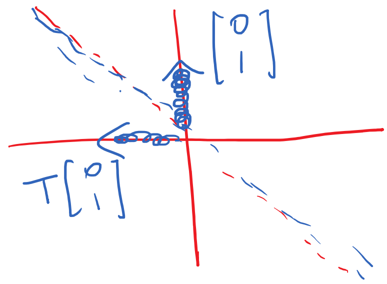
No matrix exists

No

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$

T first reflects a vector in $y=-x$ and then rotates it by 30° clockwise.

Find the standard matrix for T.



$$[T_1] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{array}{cc} T_1\begin{bmatrix}1\\0\end{bmatrix} & T_1\begin{bmatrix}0\\1\end{bmatrix} \end{array}$$

$$[T_2] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \theta = -30^\circ$$

$$= \frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}$$

$$[T] = [T_2][T_1]$$

order matters!

$$= \frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$$

Follow-Up: Image of $\begin{bmatrix} 5 \\ -7 \end{bmatrix}$ under T?

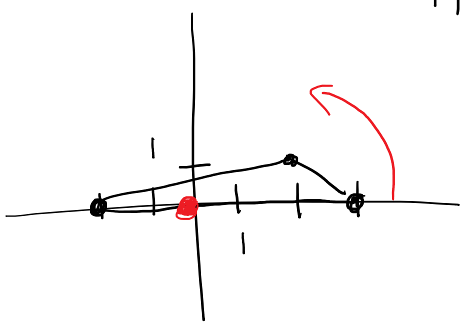
$$T\begin{bmatrix} 5 \\ -7 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}\begin{bmatrix} 5 \\ -7 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -5 + 7\sqrt{3} \\ -5\sqrt{3} - 7 \end{bmatrix}$$

output
"image"

Ex: A triangle has vertices $(-2,0)$, $(3,0)$ and $(2,1)$. Rotate the triangle by $45°$ (about the origin). Find the new vertices.



$$[T] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \theta = 45°$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$T\begin{bmatrix} -2 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$T\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$T\begin{bmatrix}3\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}3\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}3\\3\end{bmatrix}$$

$$T\begin{bmatrix}2\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\3\end{bmatrix}$$

$$\frac{1}{\sqrt{2}} \approx 0.7$$