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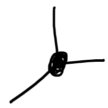
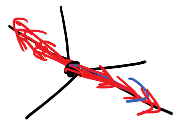
3.5 Subspaces

Some examples of subspaces of $\mathbb{R}^n$:

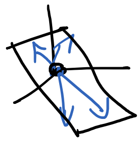
a) the origin $\{\vec{0}\}$



b) line through origin



c) plane through origin



Fact

Set S is a subspace of $\mathbb{R}^n$ if and only if

$S =$ span (one or more vectors)

Ex: Subspace of $\mathbb{R}^3$?

a) $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ such that } x+2y+3z=0 \right\}$

$x = -2y - 3z$

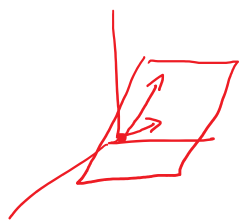
$S = \left\{ \begin{bmatrix} -2y-3z \\ y \\ z \end{bmatrix} \right\}$

$S = \left\{ \begin{bmatrix} -2y \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} -3z \\ 0 \\ z \end{bmatrix} \right\}$

$S = \left\{ y\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

$S = \text{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right)$

YES

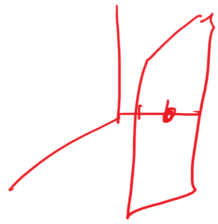


b) $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ such that } y = x + z + 2 \right\}$

$S = \left\{ \begin{bmatrix} x \\ x+z+2 \\ z \end{bmatrix} \right\}$

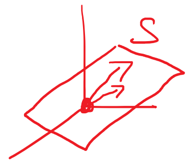
$\neq \left\{ x \begin{bmatrix} \ \end{bmatrix} + z \begin{bmatrix} \ \end{bmatrix} \right\}$

$\boxed{\text{NO}}$



Ex: Consider the subspace of $\mathbb{R}^3$:

$S = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right)$



Is the set $\mathcal{B}$ a basis for S?

a) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$

Need 2 conditions:

1) $\text{span}(\mathcal{B}) = S$ ✓ (correct direction vectors)

2) $\mathcal{B}$ is linearly independent ✓

YES

b) $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 7 \\ 14 \end{bmatrix} \right\}$

YES

c) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} \right\}$

NO

c) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} \right\}$

NO

d) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$

NO $\quad$ $\text{span}(\mathcal{B}) \neq S$

(not enough basis vectors)

Intuition: A basis is a linearly independent set of direction vectors.

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3 Subspaces Associated with a Matrix

$\text{row}(A)$ = span of rows

$\text{col}(A)$ = " columns

$\text{null}(A) = \{ \vec{x} \text{ such that } A\vec{x} = \vec{0} \}$

Ex: $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 8 & 9 \\ 3 & 10 & 3 \end{bmatrix}$

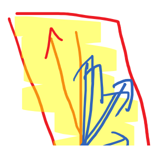
Find a basis for:

a) $\text{row}(A)$

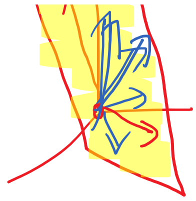
$\boxed{\text{Nonzero rows of REF/RREF}}$

⋮

RREF of $A = \begin{bmatrix} 1 & 0 & 11 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$



Basis for row(A) = { [1 0 11], [0 1 -3] }



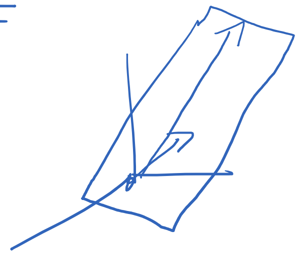
b) Col(A)

$\boxed{\text{Columns of A corresponding to the pivots}}$

$$\text{RREF of } A = \begin{bmatrix} 1 & 0 & 11 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Cols 1 and 2 of A

Basis for col(A) = $\left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} \right\}$



c) row(A), consisting of rows of A

Find a basis for Col(A^T)

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 8 & 9 \\ 3 & 10 & 3 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 8 & 10 \\ 5 & 9 & 3 \end{bmatrix}$$

$\vdots$

$$\text{RREF of } A^T = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Use Cols 1 and 2 of A^T

Basis for row(A) = $\left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 9 \end{bmatrix} \right\}$

or $\{ [1 \ 2 \ 5], [3 \ 8 \ 9] \}$

Same plane as part a)

Ex: $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 8 & 9 \\ 3 & 10 & 3 \end{bmatrix}$

Find a basis for null(A)

$\boxed{\begin{array}{l} \text{Solve } A\vec{x} = \vec{0} \\ \text{Basis} = \{\text{direction vectors}\} \end{array}}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 3 & 8 & 9 & 0 \\ 3 & 10 & 3 & 0 \end{array}\right]$$

$\vdots$

$$\left[\begin{array}{ccc|c} 1 & 0 & 11 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$z = t$

$x = -11t$

$y = 3t$

$$\vec{x} = \begin{bmatrix} -11 \\ 3 \\ 1 \end{bmatrix} t$$

Basis = $\left\{ \begin{bmatrix} -11 \\ 3 \\ 1 \end{bmatrix} \right\}$

Ex: Find a basis for span$\left(\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -4 \\ 12 \end{bmatrix} \right)$

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & 0 & 4 & 0 \\ 1 & 1 & 1 & 3 \\ 0 & 4 & -4 & 12 \end{bmatrix}$$

Find a basis for row(A):

$$\text{RREF of } A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Basis} = \{ [1 \ 0 \ 2 \ 0], [0 \ 1 \ -1 \ 3] \}$$