5.1 Orthogonality in $\mathbb{R}^n$

Orthogonal set $\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} \right\}$

Orthonormal set $\left\{ \frac{1}{\sqrt{11}}\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{22}}\begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} \right\}$

Conceptual Def

> An orthogonal matrix is a square matrix whose columns form an orthonormal set.

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Algebra Def

> A square matrix Q is orthogonal if $Q^TQ = I$

Ex: $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.6 & -0.8 \\ 0 & 0.8 & 0.6 \end{bmatrix}$

Confirm $Q^TQ = I$

$$Q^TQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.6 & 0.8 \\ 0 & -0.8 & 0.6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.6 & -0.8 \\ 0 & 0.8 & 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

1's ensure vectors have length 1

0's " are orthogonal

$\underline{Ex:}$ $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$ is orthogonal.

Find Q^{-1}.

Q orthogonal $\Rightarrow$ $Q^{-1} = Q^T$

$$Q^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 0 & -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

Why We Care about Orthogonal Matrices

(Preview) Section 5.4 Orthogonal Diagonalization

$\underline{Ex:}$ P diagonalizes A to produce D

P happens to be orthogonal.

Find a formula for A^n

$$P^{-1}AP = D$$

$$A = PDP^{-1}$$

$$A^n = PD\cancel{P^{-1}P}D\cancel{P^{-1}} \cdots \cancel{P}DP^{-1}$$

$$A^n = PD^nP^{-1}$$

$$A^n = QD^nQ^{-1}$$

$\boxed{Q^{-1} = Q^T \text{ if } Q \text{ is orthogonal}}$

$$A^n = QD^nQ^T$$

Ex: Let Q be orthogonal.
Confirm that Q^T is orthogonal.

To show M is orthogonal: $M^T M = I$

Q^T " : $(Q^T)^T Q^T = I$

$$(Q^T)^T Q^T = Q Q^T$$
$$= Q \cancel{Q^T} Q^{-1}$$
$$= I \checkmark$$

Ex: Find a, b, c so that

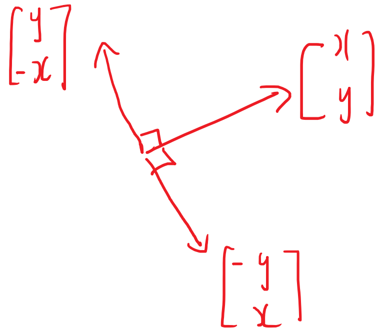
$$\begin{bmatrix} \frac{1}{3} & a \\ b & c \end{bmatrix}$$ is orthogonal.

1) $\left\| \begin{bmatrix} \frac{1}{3} \\ b \end{bmatrix} \right\| = 1$

$$\sqrt{\frac{1}{9} + b^2} = 1$$
$$\frac{1}{9} + b^2 = 1$$
$$b^2 = \frac{8}{9}$$
$$b = \pm \frac{2\sqrt{2}}{3}$$

2) If 1^{st} column is $\begin{bmatrix} x \\ y \end{bmatrix}$

then there are 2 options for 2^{nd} column:



$b = \frac{2\sqrt{2}}{3}$: $\begin{bmatrix} \frac{1}{3} & -\frac{2\sqrt{2}}{3} \\ \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix}$ $\begin{bmatrix} \frac{1}{3} & \frac{2\sqrt{2}}{3} \\ \frac{2\sqrt{2}}{3} & -\frac{1}{3} \end{bmatrix}$

$b = \frac{-2\sqrt{2}}{3}$: $\begin{bmatrix} \frac{1}{3} & \frac{2\sqrt{2}}{3} \\ -\frac{2\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix}$ $\begin{bmatrix} \frac{1}{3} & -\frac{2\sqrt{2}}{3} \\ -\frac{2\sqrt{2}}{3} & -\frac{1}{3} \end{bmatrix}$

5.1 #23: Let Q be orthogonal.
Show that $\det Q = \pm 1$.

$$Q^T Q = I$$
$$\det(Q^T Q) = \det I$$
$$\cancel{\det(Q^T)} \det Q = 1$$
$$\det Q$$
$$(\det Q)^2 = 1$$

$\det Q = \pm 1$

Ex: Consider $\bar{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and the orthogonal basis for $\mathbb{R}^3$: $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

Find the coordinate vector $[\bar{w}]_{\mathcal{B}}$

$$[\bar{w}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

where $\bar{w} = c_1 \bar{b}_1 + c_2 \bar{b}_2 + c_3 \bar{b}_3$

Orthogonal basis $\Rightarrow$ $\bar{w} = \text{proj}_{\bar{b}_1} \bar{w} + \text{proj}_{\bar{b}_2} \bar{w} + \text{proj}_{\bar{b}_3} \bar{w}$

$$\bar{w} = \frac{\vec{b}_1 \cdot \bar{w}}{\|\bar{b}_1\|^2} \bar{b}_1 + \dots$$

$$= \frac{\cancel{6}}{\cancel{3}}_{2} \bar{b}_1 + \frac{3}{\cancel{6}}_{\frac{1}{2}} \bar{b}_2 - \frac{1}{2} \bar{b}_3$$

$$[\vec{w}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ \frac{1}{2} \\ -1/2 \end{bmatrix}$$