4.2 Determinants

$$\begin{vmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & c \end{vmatrix} = abc$$

Ex: If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 7$, find:

a) $\begin{vmatrix} c & d \\ a & b \end{vmatrix} = -7$

b) $\begin{vmatrix} 4a & 4b \\ c & d \end{vmatrix} = 28$

c) $\begin{vmatrix} a & b \\ c-2a & d-2b \end{vmatrix} = 7$

Ex: A and B are 10×10 matrices.
Given $\det A = 2$ and $\det B = 3$. Find:

a) $\det A^{-1}$
$= \frac{1}{2}$

b) $\det(AB)$
$= 2 \cdot 3$
$= 6$

c) $\det(7A)$
$= 7^{10}(2)$

d) $\det B^T$
$= \det B$
$= 3$

e) $\det(6B^{-1}A^T)$
$= 6^{10} \det(B^{-1}A^T)$
$= 6^{10} \det B^{-1} \det A^T$
$= 6^{10}\left(\frac{1}{3}\right)(2)$
$= \frac{2}{3}(6^{10})$

Ex: Solve using Cramer's Rule

$$\begin{cases} 2x - 5y + z = -17 \\ 8x + y + z = -23 \\ x + 2y + 3z = 20 \end{cases}$$

i^{th} variable $= \dfrac{|A_i|}{|A|}$ $\quad (|A| \neq 0)$

where $A_i = A$, with i^{th} column replaced by $\vec{b}$

$$|A| = \begin{vmatrix} 2 & -5 & 1 \\ 8 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 132$$

$$|A_1| = \begin{vmatrix} -17 & -5 & 1 \\ -23 & 1 & 1 \\ 20 & 2 & 3 \end{vmatrix} = -528$$

cofactor expansion/ quick method

$$|A_2| = \begin{vmatrix} 2 & -17 & 1 \\ 8 & -23 & 1 \\ 1 & 20 & 3 \end{vmatrix} = 396$$

$$|A_3| = \begin{vmatrix} 2 & -5 & -17 \\ 8 & 1 & -23 \\ 1 & 2 & 20 \end{vmatrix} = 792$$

$$x = \frac{|A_1|}{|A|} = -4 \qquad y = \frac{|A_2|}{|A|} = 3 \qquad z = \frac{|A_3|}{|A|} = 6$$

Ex:

$$A = \begin{bmatrix} 2 & -5 & 1 \\ 8 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

Find the cofactors C_{11}, C_{12} and C_{32}.

Cofactor: Signed determinant associated with a matrix entry

$$\begin{bmatrix} + & - & + \\ - & + & \\ + & - & \ddots \end{bmatrix}$$

$C_{11} = +\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1 \qquad C_{12} = -\begin{vmatrix} 8 & 1 \\ 1 & 3 \end{vmatrix} = -23$

$C_{32} = -\begin{vmatrix} 2 & 1 \\ 8 & 1 \end{vmatrix} = 6$

<u>Fact</u>: $A^{-1} = \frac{1}{|A|}\operatorname{adj}(A) \qquad (|A| \neq 0)$

where $\operatorname{adj}(A) = (\text{cofactor matrix})^T$

<u>Ex</u>: Find A^{-1} using the adjoint method

$$A = \begin{bmatrix} 2 & -5 & 1 \\ 8 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$|A| = 132$ (see above)

cofactor matrix $C = \begin{bmatrix} C_{11} & C_{12} & \\ C_{21} & & \\ & \cdots & \end{bmatrix} = \begin{bmatrix} 1 & -23 & 15 \\ 17 & 5 & -9 \\ -6 & 6 & 42 \end{bmatrix}$

$\operatorname{adj}(A) = C^T = \begin{bmatrix} 1 & 17 & -6 \\ -23 & 5 & 6 \\ 15 & -9 & 42 \end{bmatrix}$

$A^{-1} = \frac{1}{132}\begin{bmatrix} 1 & 17 & -6 \\ -23 & 5 & 6 \\ 15 & -9 & 42 \end{bmatrix}$

4.2 #45

For which k is A invertible?

$$A = \begin{bmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{bmatrix} \qquad \begin{bmatrix} + & - & \\ - & & \\ + & & \end{bmatrix}$$

$$|A| = k \begin{vmatrix} k+1 & 1 \\ -8 & k-1 \end{vmatrix} + k \begin{vmatrix} -k & 3 \\ k+1 & 1 \end{vmatrix}$$

$$= k\left[(k+1)(k-1) + 8\right] + k\left[-k - 3(k+1)\right]$$

$$= \underline{\underline{k}}\left[k^2 + 7\right] + \underline{\underline{k}}\left[-4k - 3\right]$$

$$= k\left[k^2 + 7 - 4k - 3\right]$$

$$= k\left[k^2 - 4k + 4\right]$$

$$= k(k-2)^2$$

A invertible

$\iff |A| \neq 0$

$k \neq 0, 2$

4.2 #55

If $A^2 = A$, find all possible values of $\det A$

$$A^2 = A$$

$$\det(A^2) = \det A$$

$$\boxed{\det(AB) = \det A (\det B)}$$

$$[\det A]^2 = \det A$$

Let $\det A = x$

$$x^2 = x$$

$$x^2 - x = 0$$

$$x(x-1) = 0$$

$$x = 0, 1$$

Find eigenvalues of A

$$|A - \lambda I| = 0$$

Find eigenvectors of A

$$[A - \lambda I \mid \vec{0}]$$