251DX02-Mar12

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4.1 Eigenvalues and Eigenvectors

Ex: Find all eigenvalues of $A = \begin{bmatrix} 5 & -2 \\ 5 & -6 \end{bmatrix}$

Solve $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 5-\lambda & -2 \\ 5 & -6-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(-6-\lambda) + 10 = 0$$

$$\lambda^2 + \lambda - 20 = 0$$

$$(\lambda + 5)(\lambda - 4) = 0$$

$$\lambda = -5, 4$$

Ex: Find all eigenvectors of A corresponding to $\lambda = -5$

Solve $[A - \lambda I \mid \vec{0}]$

$$[A + 5I \mid \vec{0}]$$

$$\left[\begin{array}{cc|c} 10 & -2 & 0 \\ 5 & -1 & 0 \end{array}\right]$$

$$\vdots$$

$$\left[\begin{array}{cc|c} 1 & -\frac{1}{5} & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \text{RREF}$$

$x_2 = t$

$$x_1 - \frac{1}{5} x_2 = 0 \rightarrow x_1 = \frac{1}{5} t$$

$$\bar{x} = \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} t \qquad (\bar{x} \neq \bar{0})$$

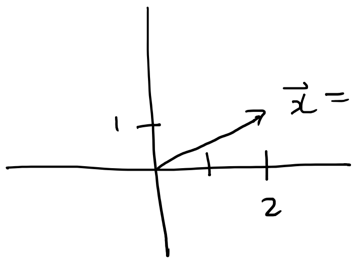
or $$\bar{x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} t \qquad (\bar{x} \neq \bar{0})$$

<u>Check</u>: $$A \begin{bmatrix} 1 \\ 5 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ 5 \end{bmatrix} \checkmark$$

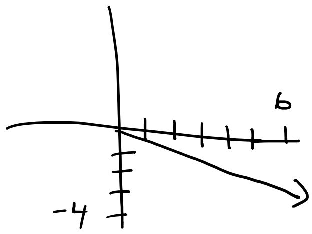
$$(A\bar{x} = \lambda \bar{x})$$

<u>Ex</u>: $$A = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$$

Find the eigenvectors and eigenvalues geometrically.



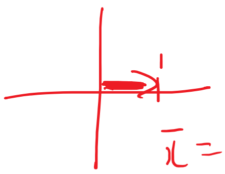
$\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$



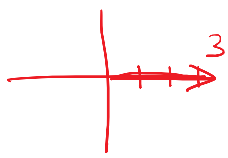
$A\vec{x} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$

Find $\bar{x}$ so that $A\bar{x}$ and $\bar{x}$ are multiples

1)



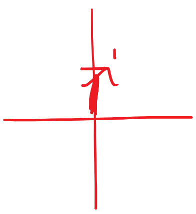
$\bar{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



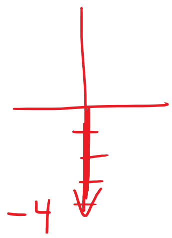
$A\bar{x} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$\lambda = 3 \qquad E_3 = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

2)



$\bar{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



$A\bar{x} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$

2)

$\lambda = -4$ $\qquad \vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} t$

$E_{-4} = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

4.1 #37 $\qquad A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \qquad (a \neq d)$

Find E_a and E_d

E_a : Solve $[A - aI \mid \vec{0}]$

$$\left[\begin{array}{cc|c} 0 & b & 0 \\ 0 & d-a & 0 \end{array}\right]$$

$\vdots$

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \text{ RREF}$$

$x_1 = t$

$x_2 = 0$

$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t \qquad (\vec{x} \neq \vec{0})$

or $\qquad E_a = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

or $\qquad$ Basis for $E_a = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

E_d : Solve $[A - \lambda I \mid \vec{0}]$

$[A - dI \mid \vec{0}]$

$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

$[a-d \quad b \mid 0]$

$$\left[\begin{array}{cc|c} a-d & b & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \text{REF}$$

$x_2 = t$

$$(a-d)x_1 + bt = 0$$

$$x_1 = \frac{-b}{a-d}\,t$$

$$= \frac{b}{d-a}\,t$$

$$\bar{x} = \begin{bmatrix} \frac{b}{d-a} \\ 1 \end{bmatrix} t \qquad (\bar{x} \neq \bar{0})$$

$$\bar{x} = \begin{bmatrix} b \\ d-a \end{bmatrix} t \qquad (\bar{x} \neq \bar{0})$$

or $\quad E_d = \text{span}\left(\begin{bmatrix} b \\ d-a \end{bmatrix} \right)$

or $\quad$ Basis for $E_d = \left\{ \begin{bmatrix} b \\ d-a \end{bmatrix} \right\}$

4.2 Determinants

- Cofactor expansion
- Quick Method (only for 3x3)
- Row operations

Ex: Find det A by cofactor expansion

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 8 \\ 3 & 1 & 2 & 6 \\ 2 & 1 & 2 & 1 \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - \\ - & + & & \\ & & \ddots & \\ & & & + \end{bmatrix}$$

$$|A| = 8\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 1 & 2 \end{vmatrix} \qquad \begin{bmatrix} + & - & + \\ & \cdots & \\ & & + \end{bmatrix}$$

$$= 8\left[1\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix}\right]$$

$$= 8\left[\,1(0) - 2(2) + 3(1)\,\right]$$

$$= -8$$

Ex: Find $|A|$ using the quick method

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 8 \end{bmatrix}$$

$$\begin{array}{ccccc} \oplus & \oplus & \oplus & \ominus & \ominus & \ominus \\ 1 & 2 & 3 & 1 & 2 & \\ 4 & 5 & 6 & 4 & 5 & \\ 1 & 2 & 8 & 1 & 2 & \\ -15 & -12 & -64 & 40 & 12 & 24 \end{array}$$

$$|A| = -15 - 12 - 64 + 40 + 12 + 24$$

$$= -15$$

Fact: If A is upper/lower triangular or diagonal then determinant = product of diagonal entries.

$$\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = 3$$

$$\begin{vmatrix} 1 & 0 & 0 \\ x & 2 & 0 \\ y & z & 3 \end{vmatrix} = 6$$

$$\begin{vmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix} = 48$$

Ex: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has $\det A = 12$

Find the determinant of:

a) $B = \begin{bmatrix} a & b \\ 3c & 3d \end{bmatrix}$ $\quad \det B = 3 \cdot 12 = 36$

b) $B = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$ $\quad \det B = 3 \cdot 3 \cdot 12 = 108$

c) $B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$ $\quad \det B = -12$

d) $B = \begin{bmatrix} a & b \\ c+7a & d+7b \end{bmatrix}$ $\quad \det B = 12$ (no change)