251DX02Feb26

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3.4 LU Factorization

Ex: Find the LU factorization for $A = \begin{bmatrix} 4 & 8 & 12 \\ 2 & -2 & 24 \\ -12 & -28 & -19 \end{bmatrix}$

A → REF using: (current row) − k(pivot row)

$R_2 - \frac{1}{2}R_1$
$R_3 + 3R_1$ $\begin{bmatrix} 4 & 8 & 12 \\ 0 & -6 & 18 \\ 0 & -4 & 17 \end{bmatrix}$ $k = \frac{1}{2}$, $k = -3$

$\boxed{\begin{gathered} 2 - k(4) = 0 \\ k = \frac{1}{2} \end{gathered}}$

$R_3 - \frac{2}{3}R_2$ $\begin{bmatrix} 4 & 8 & 12 \\ 0 & -6 & 18 \\ 0 & 0 & 5 \end{bmatrix}$ $k = \frac{2}{3}$

$\boxed{\begin{gathered} -4 - k(-6) = 0 \\ k = \frac{2}{3} \end{gathered}}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -3 & \frac{2}{3} & 1 \end{bmatrix}$ $U = \begin{bmatrix} 4 & 8 & 12 \\ 0 & -6 & 18 \\ 0 & 0 & 5 \end{bmatrix}$

(the REF)

Why This Works:

$R_2 - \frac{1}{2}R_1$ $E = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

L <u>undoes</u> the operations that turn A into U

L turns U into A

$LU = A$

Ex: Solve $A\vec{x} = \begin{bmatrix} 28 \\ 14 \\ -79 \end{bmatrix}$

using $A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -3 & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 4 & 8 & 12 \\ 0 & -6 & 18 \\ 0 & 0 & 5 \end{bmatrix}$

$$L \underbrace{\cancel{U\vec{x}}}_{\vec{y}} = \vec{b}$$

1) $L\vec{y} = \vec{b}$

$$\begin{array}{c} \begin{matrix} y_1 & y_2 & y_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 28 \\ \frac{1}{2} & 1 & 0 & 14 \\ -3 & \frac{2}{3} & 1 & -79 \end{array}\right] \end{array}$$

$$\boxed{y_1 = 28}$$

$\frac{1}{2}y_1 + y_2 = 14 \rightarrow 14 + y_2 = 14 \rightarrow \boxed{y_2 = 0}$

$-3y_1 + \frac{2}{3}y_2 + y_3 = -79 \rightarrow -84 + y_3 = -79 \rightarrow \boxed{y_3 = 5}$

2) $U\vec{x} = \vec{y}$

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} 4 & 8 & 12 & 28 \\ 0 & -6 & 18 & 0 \\ 0 & 0 & 5 & 5 \end{array}\right] \end{array}$$

$5x_3 = 5 \rightarrow \boxed{x_3 = 1}$

$-6x_2 + 18x_3 = 0 \rightarrow -6x_2 + 18 = 0 \rightarrow \boxed{x_2 = 3}$

$4x_1 + 8x_2 + 12x_3 = 28 \rightarrow 4x_1 + 24 + 12 = 28 \rightarrow \boxed{x_1 = -2}$

$$\vec{x} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

Test 2.3 - 3.4

5 Questions

45 mins to solve / 15 mins to upload

Allowed Course website / your personal notes

Ex: a) Is $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ a linear combination of $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$?

b) Are the 3 vectors linearly independent?

a) Let $c_1 \vec{v} + c_2 \vec{w} = \vec{u}$

System is consistent (solvable) → YES

System is inconsistent → NO

$$\begin{array}{cc} c_1 & c_2 \\ \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{array}\right]$$

$$\begin{array}{l} R_2 - R_1 \\ R_3 - 2R_1 \end{array} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & -3 & 1 \end{array}\right]$$

$$R_3 - 3R_2 \quad \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array}\right] \quad \text{REF}$$

System has no solution.

$\boxed{\text{NO}}$

b) Let $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{0}$

Unique solution $c_1 = c_2 = c_3 = 0$ → $\boxed{\text{LINEARLY INDEPENDENT}}$

Infinitely-Many Solutions! → $\boxed{\text{LIN. DEPENDENT}}$

$$\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{l} R_2 - R_1 \\ R_3 - 3R_1 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & -5 & 0 \end{array}\right]$$

$$R_2 \Leftrightarrow R_3 \quad \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -1 & -5 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \end{array} \quad \text{REF}$$

1 solution

$c_1 = c_2 = c_3 = 0$

$\boxed{\text{YES}}$

Ex: Labour Hours Required

$$\begin{array}{ccc} & \text{Shift } 1 & 2 \\ \text{Site } A & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{bmatrix} 20 & 15 \\ 12 & 15 \end{bmatrix}$}} \\ B & & \end{array}$$

Labour Cost ($/hr)

$$\begin{array}{cc} \text{Shift } 1 & 2 \\ \multicolumn{2}{c}{\begin{bmatrix} 12 & 13 \end{bmatrix}} \end{array}$$

Compute total labour cost at each site.

$$\begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 20 & 15 \\ 12 & 15 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array}\begin{bmatrix} 12 \\ 13 \end{bmatrix} = \begin{array}{c} A \\ B \end{array}\begin{bmatrix} 435 \\ 339 \end{bmatrix}$$