251DX02Feb23

February 23, 2021 3:28 PM

3.3 Elementary Matrices

Elementary matrix: represents a row operation

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ c & d \end{bmatrix}$$

$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ represents $2R_1 \begin{bmatrix} \ \end{bmatrix}$

Key Fact: Elementary matrices act on the **left**

Ex: a) Find the elementary matrix for

$R_2 - 3R_1 \begin{bmatrix} \ \end{bmatrix}$

How is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ changing?

$$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

b) Find the E. M. for

$R_1 \leftrightarrow R_2 \begin{bmatrix} \ \end{bmatrix}$

How is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ changing?

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

c) Which operation is encoded by $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$?

How is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ changing?

$\begin{bmatrix} \ \end{bmatrix}$

$R_2+4R_1 \begin{bmatrix} & \\ & \end{bmatrix}$

d) Which operation is encoded by $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$?

$\frac{R_1}{2} \begin{bmatrix} & \\ & \end{bmatrix}$

e.g. $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{a}{2} & \frac{b}{2} \\ c & d \end{bmatrix}$

Ex: Write the inverse operation

a) $R_2+3R_1 \begin{bmatrix} & \\ & \end{bmatrix}$

$R_2-3R_1 \begin{bmatrix} & \\ & \end{bmatrix}$

b) $R_1 \Leftrightarrow R_2$

$R_1 \Leftrightarrow R_2$

c) $\frac{R_1}{4} \begin{bmatrix} & \\ & \end{bmatrix}$

$4R_1 \begin{bmatrix} & \\ & \end{bmatrix}$

Fact: If E encodes a row operation
then E^{-1} " the inverse operation

Ex:

Operation	Elementary Matrix E	Inverse Operation	E^{-1}
$R_2 - 2R_1$	$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$	$R_2 + 2R_1$	$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$
$R_1 \Leftrightarrow R_2$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$R_1 \Leftrightarrow R_2$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
$\frac{R_2}{5}$	$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$	$5R_2$	$\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$

Ex: $A = \begin{bmatrix} 3 & 6 \\ 1 & 1 \end{bmatrix}$

Write A and A^{-1} as a product of elementary matrices.

$$\begin{bmatrix} 3 & 6 \\ 1 & 1 \end{bmatrix}$$

$R_1 \Leftrightarrow R_2$ $\begin{bmatrix} 1 & 1 \\ 3 & 6 \end{bmatrix}$ $\quad E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad E_1^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $(R_1 \Leftrightarrow R_2)$

$R_2 - 3R_1$ $\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ $\quad E_2 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ $\quad E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ $(R_2 + 3R_1)$

$\frac{R_2}{3}$ $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$ $\quad E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ $(3R_2)$

$R_1 - R_2$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad E_4 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ $\quad E_4^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $(R_1 + R_2)$

$$\underbrace{E_4 E_3 E_2 E_1}_{A^{-1}} A = I$$

$$A^{-1} = E_4 E_3 E_2 E_1$$

$$A = (A^{-1})^{-1} = (E_4 E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$$ ✓

order reverses

Fundamental Theorem of Invertible Matrices

Let A be an $n \times n$ matrix.
The following are all true or all false.

i) A^{-1} exists

ii) $A\vec{x} = \vec{b}$ has a unique solution.

iii) The RREF of A is I

iv) A is a product of elementary matrices

v) The only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$
very similar to ii)

Visually:

$\boxed{A^{-1} \text{ exists}} \longleftrightarrow \boxed{\vec{x} = A^{-1}\vec{b}} \longleftrightarrow \boxed{\begin{matrix}[A \mid \vec{b}] \\ \vdots \\ [I \mid \#]\end{matrix}} \longleftrightarrow \boxed{\begin{matrix}A \text{ is a} \\ \text{product of} \\ \text{E.M.}\end{matrix}}$

Ex: Does
$$\begin{cases} 3x + 5y = 7 \\ 6x - 11y = 9 \end{cases}$$
have a unique solution?

$$A = \begin{bmatrix} 3 & 5 \\ 6 & -11 \end{bmatrix}$$

$\det A \neq 0$

A^{-1} exists

$\boxed{\text{YES}}$

Ex: Is $A = \begin{bmatrix} 1 & 3 \\ 4 & 12 \end{bmatrix}$ a product of elementary matrices?

$\det A = 0$

A^{-1} d.n.e.

$\boxed{\text{No}}$