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February 22, 2021 3:29 PM

3.3 The Inverse of a Matrix

Ex: Find A^{-1}

a) $A = \begin{bmatrix} 7 & -2 \\ 3 & 6 \end{bmatrix}$

$\det A = 48$

$A^{-1} = \frac{1}{48}\begin{bmatrix} 6 & 2 \\ -3 & 7 \end{bmatrix}$

b) $A = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$

$\det A = 0$

A^{-1} does not exist

A^{-1} is undefined

A is not invertible

Ex: Use A^{-1} to solve

$$\begin{cases} x - 3y = 27 \\ 4x + 5y = -11 \end{cases}$$

$A = \begin{bmatrix} 1 & -3 \\ 4 & 5 \end{bmatrix}$

$A^{-1} = \frac{1}{17}\begin{bmatrix} 5 & 3 \\ -4 & 1 \end{bmatrix}$

$\circledast$ $\vec{x} = A^{-1}\vec{b}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 5 & 3 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 27 \\ -11 \end{bmatrix}$

$$= \frac{1}{17}\begin{bmatrix} 102 \\ -119 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -7 \end{bmatrix}$$

<u>Ex</u>: Find A^{-1} for $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$$[A \mid I] \longrightarrow [I \mid A^{-1}]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$R_2 - 2R_1 \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\frac{R_2}{-1} \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_1 - R_2 \\ \\ R_3 - R_2 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 2 & -2 & 1 & 1 \end{array}\right]$$

$$\frac{R_3}{2} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\begin{array}{l} R_1 - R_3 \\ R_2 + R_3 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -1 & \frac{1}{2} & \frac{1}{2} \end{array}\right] \quad = A^{-1}$$

Motivation: Can solve $A\bar{x} = \bar{b}$
using $\bar{x} = A^{-1}\bar{b}$

Ex: $B = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ Find B^{-2}

$$B^{-2} = (B^{-1})^2 \quad \text{or} \quad (B^2)^{-1}$$

$$B^2 = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 9 \\ 12 & 16 \end{bmatrix}$$

$$B^{-2} = (B^2)^{-1} = \frac{1}{100}\begin{bmatrix} 16 & -9 \\ -12 & 13 \end{bmatrix}$$

Ex: Solve for the invertible matrix X given:

$$X^{-1} = A^{-1}B^T$$

$$(X^{-1})^{-1} = (A^{-1}B^T)^{-1}$$

$$X = (A^{-1}B^T)^{-1}$$

$$X = (B^T)^{-1}(A^{-1})^{-1}$$

$$X = (B^T)^{-1}A$$

or $$X = (B^{-1})^T A$$

3 Rules:

$$(A^{-1})^{-1} = A$$

$$(A^T)^{-1} = (A^{-1})^T$$

$$(AB)^{-1} = B^{-1}A^{-1} \quad *$$

Ex: Find 2x2 matrices A and B so that

$$(A-B)^{-1} \neq A^{-1} - B^{-1}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(A-B)^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^{-1} = \text{undefined}$$

$$A^{-1} - B^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Many examples exist

Ex: Find nonzero 2x2 matrices A, B and C so that

$$AB = AC \quad \text{but} \quad B \neq C$$

Observation: A can't be invertible

Sol #1 $\quad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = AC$$

Sol #2 $\quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = AC$$
