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3.2 Matrix Algebra

Ex: Is $\begin{bmatrix} -1 & -14 \\ -23 & 3 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$?

Let $c_1 \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -14 \\ -23 & 3 \end{bmatrix}$

System is solvable → YES

System has no solution → NO

$$\begin{array}{cc} c_1 & c_2 \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 1 & -1 \\ 2 & 4 & -14 \\ -1 & 3 & -23 \\ 3 & 2 & 3 \end{array}\right]$$

$R_2 - 2R_1$, $R_3 + R_1$, $R_4 - 3R_1$

$$\left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 2 & -12 \\ 0 & 4 & -24 \\ 0 & -1 & 6 \end{array}\right]$$

$\frac{R_2}{2}$

$$\left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & -6 \\ 0 & 4 & -24 \\ 0 & -1 & 6 \end{array}\right]$$

$R_3 - 4R_2$, $R_4 + R_2$

$$\begin{array}{cc} c_1 & c_2 \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \text{REF}$$

System is consistent (solvable)

$\boxed{\text{YES}}$

Side note: A system has no solution exactly when the REF/RREF has a row like $\left[\begin{array}{cc|c} 0 & 0 & \text{nonzero} \end{array}\right]$

Ex: Find the general form of
$\text{span}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\right)$

$\longrightarrow \quad c_1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$

$\longrightarrow$ Each zero row of the REF leads to a condition on w, x, y, z

$$\begin{array}{ccc|c} c_1 & c_2 & c_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & w \\ 0 & 1 & -1 & x \\ 0 & 1 & 1 & y \\ 1 & 0 & 1 & z \end{array}\right]$$

$$R_4 - R_1 \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & w \\ 0 & 1 & -1 & x \\ 0 & 1 & 1 & y \\ 0 & 0 & 0 & z-w \end{array}\right]$$

$\vdots$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & w \\ 0 & 1 & -1 & x \\ 0 & 0 & 2 & y-x \\ 0 & 0 & 0 & z-w \end{array}\right] \quad \text{REF}$$

System is consistent $\Rightarrow z - w = 0$
$\Rightarrow z = w$

$$\text{span} = \left\{ \begin{bmatrix} w & x \\ y & z \end{bmatrix} \text{ such that } z = w \right\} \checkmark$$

$$= \left\{ \begin{bmatrix} w & x \\ y & w \end{bmatrix} \right\}$$

Ex: Are $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ linearly independent?

Let $c_1 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + c_2 \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$c_1 = 0,\ c_2 = 0,\ c_3 = 0$ → $\boxed{\text{Yes}}$

$c_1 = c_2 = c_3 = 0$ and other solutions → $\boxed{\text{No}}$

$$\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ 2 & 6 & 1 & 0 \\ 3 & 7 & 1 & 0 \\ 4 & 8 & 1 & 0 \end{array}\right] \end{array}$$

$\vdots$

$$\left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ 0 & 1 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{REF}$$

$c_3 = t$

∞-many solutions

$\boxed{\text{No}}$

Quick Ex:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{REF}$$

$c_1 = c_2 = c_3 = 0$

Matrices are linearly independent.

Ex: Find 2x2 matrices A and B so that A and B are symmetric but AB is not symmetric.

$\boxed{\begin{array}{l} A \text{ is symmetric if } A^T = A \\ \text{e.g. } \begin{bmatrix} 9 & 2 \\ 2 & 11 \end{bmatrix} \end{array}}$

Choose $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ $\quad B = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$

$$AB = \begin{bmatrix} x+2y & y+2z \\ 2x+3y & 2y+3z \end{bmatrix}$$

Want $y + 2z \neq 2x + 3y$

e.g. $y = 1,\ z = 0,\ x = 0$

$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

<u>Ex</u>: Simplify $A\left(A^T - B^T\right)^T I$

$$= A\left(\left(A^T\right)^T - \left(B^T\right)^T\right) I$$

$$= A\left(A - B\right) I$$

$$= \left(A^2 - AB\right) I$$

$$= A^2 - AB$$