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April 9, 2021 1:25 PM

5.4 Orthogonal Diagonalization

Ex: $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$ has $\lambda = 1, 7$

Find Q that orthogonally diagonalizes A.

Find an orthonormal basis for each eigenspace.

E_7: $[A - 7I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} -4 & 2 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ RREF}$$

$x_3 = t$

$x_1 = t$

$x_2 = t$

$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t$

Orthonormal Basis for E_7 $\left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

E_1: $[A - I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ RREF}$$

$x_2 = s$

$x_3 = t$

$x_1 = -s - t$

$\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} t$

Do Gram-Schmidt on $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Partial Basis $X = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \text{proj}_{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$2\vec{v}_2 = 2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

<u>Orthonormal</u> Basis for E_1 : $\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$$

<u>Ex</u>: Find a 3x3 matrix A with $\lambda = 2, 4$

and $E_2 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$

$E_4 = \text{span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right)$

If eigenvectors are <u>orthonormal</u>

Section 4.4

$A = PDP^{-1}$

(May be Inconvenient to find P^{-1})

Section 5.4

$$\boxed{\begin{gathered} A = \lambda_1 \vec{q}_1 \vec{q}_1^{\,T} + \dots + \lambda_n \vec{q}_n \vec{q}_n^{\,T} \\ \text{where } \vec{q}_1, \dots, \vec{q}_n \text{ are } \underline{\text{orthonormal}} \text{ eigenvectors} \end{gathered}}$$

$$\vec{q}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{q}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \vec{q}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\lambda_1 = 2 \qquad \lambda_2 = 2 \qquad \lambda_3 = 4$$

$$A = \underbrace{\cancel{2 \cdot \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}}}_{\frac{2}{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} + \underbrace{\cancel{\frac{2}{2}}}_{\frac{3}{3}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} + \underbrace{\cancel{\frac{4}{6}}}_{\frac{2}{3}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$$

$$= \frac{2}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \frac{3}{3} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 7 & -2 & 1 \\ -2 & 10 & -2 \\ 1 & -2 & 7 \end{bmatrix}$$

Exam Review

Ex: Find 3 orthogonal vectors with length 5 so that the first is parallel to [8, 7, 2] and the second has the form c[3, 8, k].

$$[8, 7, 2] \cdot [3, 8, k] = 0$$

$$24 + 56 + 2k = 0$$

$$k = -40$$

3rd direction is $[8, 7, 2] \times [3, 8, -40]$

$$= [-296, 326, 43]$$

$$\begin{matrix} 8 & 7 & 2 & 8 & 7 \\ 3 & 8 & -40 & 3 & 8 \end{matrix}$$

The 3 vectors are:

$$\frac{5}{\sqrt{117}}[8, 7, 2]$$

$$\frac{5}{\sqrt{1673}}[3, 8, -40]$$

$$\frac{5}{\sqrt{195741}}[-296, 326, 43]$$

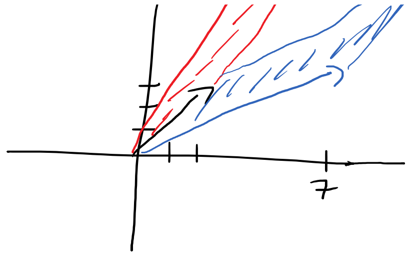
<u>Ex</u>: a) Find a so that the area of the parallelogram formed by $[2, 3]$ and $[7, a]$ is 12.

$$\left| \det \begin{bmatrix} 2 & 3 \\ 7 & a \end{bmatrix} \right| = 12$$

$$| 2a - 21 | = 12$$

$$2a - 21 = 12 \qquad\qquad 2a - 21 = -12$$

$$a = \frac{33}{2} \qquad\qquad a = \frac{9}{2}$$



b) Find the area of the parallelogram formed by $[2,3,1]$ and $[7,-2,4]$.

$$\text{Area}(\text{▱ in } \mathbb{R}^3) = \|\bar{u} \times \bar{v}\|$$

$$\begin{matrix} 2 & 3 & 1 & 2 & 3 \\ 7 & -2 & 4 & 7 & -2 \end{matrix}$$

$$\bar{u} \times \bar{v} = [14, -1, -25]$$

$$\|\bar{u} \times \bar{v}\| = \sqrt{822}$$

c) Volume of the parallelepiped formed by $[1,2,3]$, $[4,5,6]$, $[7,8,c]$?

$$\text{Volume} = \left| \det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & c \end{bmatrix} \right|$$

$$= \left| 1(5c-48) - 2(4c-42) + 3(-3) \right|$$

$$= \left| 27 - 3c \right|$$