251DX02Apr6

April 6, 2021 3:27 PM

5.3 Gram-Schmidt

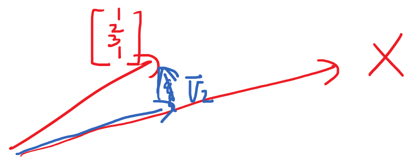
Ex: Let $W = \operatorname{span}\left(\begin{bmatrix}1\\2\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\\1\end{bmatrix}, \begin{bmatrix}2\\4\\2\\1\end{bmatrix} \right)$

Find an orthogonal basis for W.

$$\vec{v}_1 = \begin{bmatrix}1\\2\\1\\1\end{bmatrix}$$

Partial Basis $X = \left\{ \begin{bmatrix}1\\2\\1\\1\end{bmatrix} \right\}$

$$\vec{v}_2 = \begin{bmatrix}1\\2\\3\\1\end{bmatrix} - \operatorname{proj}_X \begin{bmatrix}1\\2\\3\\1\end{bmatrix}$$



$$= \begin{bmatrix}1\\2\\3\\1\end{bmatrix} - \frac{9}{7}\begin{bmatrix}1\\2\\1\\1\end{bmatrix}$$

Can scale:

$$7\vec{v}_2 = 7\begin{bmatrix}1\\2\\3\\1\end{bmatrix} - 9\begin{bmatrix}1\\2\\1\\1\end{bmatrix} = \begin{bmatrix}-2\\-4\\12\\-2\end{bmatrix}$$

Scale again: $\begin{bmatrix}1\\2\\-6\\1\end{bmatrix}$

Partial Basis $X = \left\{ \begin{bmatrix}1\\2\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\-6\\1\end{bmatrix} \right\}$

$$\vec{v}_3 = \begin{bmatrix}2\\4\\2\\1\end{bmatrix} - \operatorname{proj}_X \begin{bmatrix}2\\4\\2\\1\end{bmatrix}$$

$$= \begin{bmatrix}2\\4\\2\\1\end{bmatrix} - \operatorname{proj}_{\begin{bmatrix}1\\2\\1\\1\end{bmatrix}} \begin{bmatrix}2\\4\\2\\1\end{bmatrix} - \operatorname{proj}_{\begin{bmatrix}1\\2\\-6\\1\end{bmatrix}} \begin{bmatrix}2\\4\\2\\1\end{bmatrix}$$

$$= \begin{bmatrix}2\\4\\2\\1\end{bmatrix} - \frac{13}{7}\begin{bmatrix}1\\2\\1\\1\end{bmatrix} - \frac{(-1)}{42}\begin{bmatrix}1\\2\\-6\\1\end{bmatrix}$$

$$42\vec{v}_3 = 42\begin{bmatrix}2\\4\\2\\1\end{bmatrix} - 78\begin{bmatrix}1\\2\\1\\1\end{bmatrix} + 1\begin{bmatrix}1\\2\\-6\\1\end{bmatrix} = \begin{bmatrix}7\\14\\0\\-35\end{bmatrix}$$

Can scale: $\begin{bmatrix}1\\2\\0\\-5\end{bmatrix}$

Orthogonal Basis $= \left\{ \begin{bmatrix}1\\2\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\-6\\1\end{bmatrix}, \begin{bmatrix}1\\2\\0\\-5\end{bmatrix} \right\}$

all dot products $= 0$ ✓

Ex: Find QR Factorization for $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 1 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

orthogonal

upper triangular

Q: Gram-Schmidt on columns of A

Then normalize

$$\left\{ \frac{1}{\sqrt{7}}\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{42}}\begin{bmatrix} 1 \\ 2 \\ -6 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{30}}\begin{bmatrix} 1 \\ 2 \\ 0 \\ -5 \end{bmatrix} \right\}$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{7}} & \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{7}} & \vdots & \vdots \\ \frac{1}{\sqrt{7}} & & \\ \frac{1}{\sqrt{7}} & & \end{bmatrix}$$

$$R = Q^T A$$

$$= \begin{bmatrix} \frac{1}{\sqrt{7}} & \frac{2}{\sqrt{7}} & \frac{1}{\sqrt{7}} & \frac{1}{\sqrt{7}} \\ \frac{1}{\sqrt{42}} & \frac{2}{\sqrt{42}} & \frac{-6}{\sqrt{42}} & \frac{1}{\sqrt{42}} \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & 0 & \frac{-5}{\sqrt{30}} \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 1 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{\sqrt{7}} & \frac{9}{\sqrt{7}} & \frac{13}{\sqrt{7}} \\ 0 & \frac{-12}{\sqrt{42}} & \frac{-1}{\sqrt{42}} \\ 0 & 0 & \frac{5}{\sqrt{30}} \end{bmatrix}$$

Exam Review

Ex: Find the angle between $\bar{u} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 9 \end{bmatrix}$ and $\bar{v} = \begin{bmatrix} 4 \\ -5 \\ 4 \\ 1 \end{bmatrix}$

$$\bar{u} \cdot \bar{v} = \|\bar{u}\| \, \|\bar{v}\| \cos\theta$$

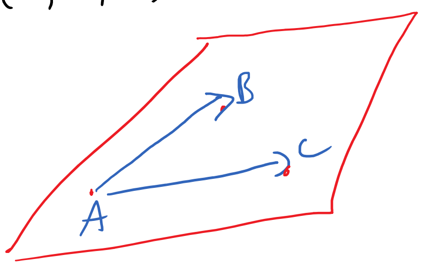
$$15 = \sqrt{95}\,\sqrt{58}\,\cos\theta$$

$$\frac{15}{\sqrt{95}\,\sqrt{58}} = \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{15}{\sqrt{95}\,\sqrt{58}}\right)$$

$$\approx 78.3^\circ$$

Ex: Find general and normal form of the plane through
$A = (1,1,1)$
$B = (3,0,4)$
$C = (2,6,5)$



$$\vec{AB} = [2, -1, 3]$$

$$\vec{AC} = [1, 5, 4]$$

$$\vec{n} = \vec{AB} \times \vec{AC}$$

$$= [-19, -5, 11]$$

$$\boxed{\begin{matrix} 2 & -1 & 3 & 2 & -1 \\ 1 & 5 & 4 & 1 & 5 \end{matrix}}$$

Normal Form

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$$

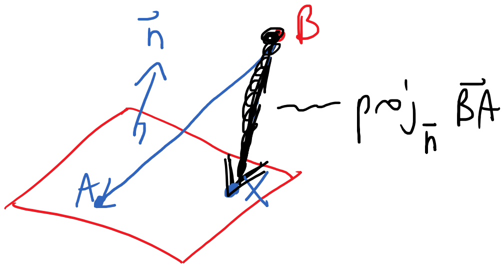
$$\begin{bmatrix} -19 \\ -5 \\ 11 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -19 \\ -5 \\ 11 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -5 \\ 11 \end{bmatrix} \cdot \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ 11 \end{bmatrix} \cdot \begin{bmatrix} \\ 1 \end{bmatrix}$$

any point on plane

General Form $\quad -19x - 5y + 11z = -13$

<u>Ex</u>: Find the point X on the plane $2x - 7y + 3z = 6$ that is closest to $B = (1, 2, 1)$.



A: any point on plane

$$A = (3, 0, 0)$$

$$\overrightarrow{BA} = [2, -2, -1] \qquad \text{Think } A - B$$

$$\vec{n} = [2, -7, 3]$$

$$\operatorname{proj}_{\vec{n}} \overrightarrow{BA} = \frac{\vec{n} \cdot \overrightarrow{BA}}{\|\vec{n}\|^2} \vec{n}$$

$$= \frac{15}{62} \begin{bmatrix} 2 \\ -7 \\ 3 \end{bmatrix}$$

$$\vec{X} = \vec{B} + \operatorname{proj}_{\vec{n}} \overrightarrow{BA} \qquad \text{(see sketch)}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \frac{15}{62} \begin{bmatrix} 2 \\ -7 \\ 3 \end{bmatrix}$$

$$\frac{1}{\ } \begin{bmatrix} 92 \\ 19 \end{bmatrix}$$

$$= \frac{1}{62}\begin{bmatrix} 92 \\ 19 \\ 107 \end{bmatrix}$$

$$X = \left(\frac{92}{62}, \frac{19}{62}, \frac{107}{62}\right)$$

PLAN:

Fri	5.4	+ Review
Mon	Complex	
Tues	Complex	
Fri	7.3	