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April 13, 2021 3:29 PM

Complex Numbers

Ex: a) Express $z = 7e^{\frac{\pi}{4}i}$ in rectangular form $(z = a+bi)$

$$z = |z|[\cos\theta + i\sin\theta]$$

$$z = 7\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$$

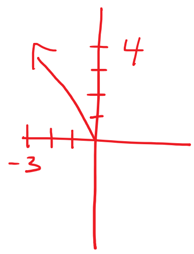
$$z = 7\left[\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right]$$

$$z = \frac{7\sqrt{2}}{2} + \frac{7\sqrt{2}}{2}i$$

b) Find the exponential form for $z = -3 + 4i$

$$|z| = \sqrt{(-3)^2 + 4^2} = 5$$

$$\theta = \tan^{-1}\left(\frac{4}{-3}\right) \quad (+\pi?)$$



$$\theta = \pi - \tan^{-1}\frac{4}{3}$$

$$z = |z|\,e^{i\theta}$$

$$z = 5\,e^{(\pi - \tan^{-1}\frac{4}{3})i}$$

Ex: $A = \begin{bmatrix} 5 & -13 \\ 5 & 3 \end{bmatrix}$

Find all eigenvalues.

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 5-\lambda & -13 \\ 5 & 3-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(3-\lambda)+65=0$$

$$\lambda^2-8\lambda+80=0$$

$$\lambda=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\lambda=\frac{8\pm\cancel{\sqrt{-256}}}{2} \quad 16i$$

$$\lambda=4\pm 8i$$

Follow-Up: Find a basis for E_{4+8i}

$$[A-\lambda I \mid \vec{0}]$$

$$[A-(4+8i)I \mid \vec{0}]$$

$$\left[\begin{array}{cc|c} 1-8i & -13 & 0 \\ 5 & -1-8i & 0 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$\frac{R_1}{5}$

$$\left[\begin{array}{cc|c} 1 & \frac{-1-8i}{5} & 0 \\ 1-8i & -13 & 0 \end{array}\right]$$

$R_2-(1-8i)R_1$ $$\left[\begin{array}{cc|c} 1 & \frac{-1-8i}{5} & 0 \\ 0 & 0 & 0 \end{array}\right]$$ RREF

Conceptually

OR

$$-13-(1-8i)\left(\frac{-1-8i}{5}\right)$$

$$=-13-\left(\frac{-1-64}{5}\right)$$

$$=0$$

$$x_2=t$$

$$x_1 = \frac{1+8i}{5} t$$

$$\bar{x} = \begin{bmatrix} \frac{1+8i}{5} \\ 1 \end{bmatrix} t \qquad (\bar{x} \neq \bar{0})$$

$$\bar{x} = \begin{bmatrix} 1+8i \\ 5 \end{bmatrix} t$$

Basis for $E_{4+8i} = \left\{ \begin{bmatrix} 1+8i \\ 5 \end{bmatrix} \right\}$

Exam Review

Ex: Find a basis for each subspace S of $\mathbb{R}^3$ and state $\dim(S)$

a) $S = \left\{ \begin{bmatrix} 4y \\ y \\ z \end{bmatrix} \right\}$

$$S = \left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} z \right\}$$

Basis for $S = \left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\dim(S) = 2$

b) $S = \left\{ \begin{bmatrix} x \\ 3x-4z \\ z \end{bmatrix} \right\}$

$$S = \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -4 \\ 1 \end{bmatrix} z \right\}$$

Basis for $S = \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 1 \end{bmatrix} \right\}$

$\dim(S) = 2$

Ex: Write $A = \begin{bmatrix} 3 & 3 \\ 1 & 4 \end{bmatrix}$ as a

$I = \begin{bmatrix} 1 & 0 \end{bmatrix}$

Ex: Write $A = \begin{bmatrix} 3 & 3 \\ 1 & 4 \end{bmatrix}$ as a product of elementary matrices.

$\boxed{I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$

$\frac{R_1}{3}$ $\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$ $E_1 = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$ $E_1^{-1} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ $(3R_1)$

$R_2 - R_1$ $\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ $E_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ $E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $(R_2 + R_1)$

$\frac{R_2}{3}$ $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$ $E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ $(3R_2)$

$R_1 - R_2$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $E_4 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ $E_4^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $(R_1 + R_2)$

$$\underbrace{E_4 E_3 E_2 E_1}_{A^{-1}} A = I$$

$$A = (A^{-1})^{-1}$$

$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$ order reverses

Ex: A is a 2×2 matrix with eigenvectors $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ corresponding to $\lambda = -2$ and $\lambda = 2$ respectively. Find and simplify A^n.

$$A^n = P D^n P^{-1}$$

$P = \begin{bmatrix} 5 & 3 \\ \ldots \end{bmatrix}$ $D = \begin{bmatrix} -2 & 0 \\ \ldots \end{bmatrix}$ $D^n = \begin{bmatrix} (-2)^n & 0 \\ \ldots \end{bmatrix}$

$$P = \begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix} \qquad D = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \qquad D^n = \begin{bmatrix} (-2)^n & 0 \\ 0 & 2^n \end{bmatrix}$$

$$P^{-1} = \frac{1}{14}\begin{bmatrix} 4 & -3 \\ -2 & 5 \end{bmatrix}$$

$$A^n = P D^n P^{-1}$$

$$= \frac{1}{14}\begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} (-2)^n & 0 \\ 0 & 2^n \end{bmatrix}\begin{bmatrix} 4 & -3 \\ -2 & 5 \end{bmatrix}$$

$$= \frac{1}{14}\begin{bmatrix} 5(-2)^n & 3\cdot 2^n \\ 2(-2)^n & 4\cdot 2^n \end{bmatrix}\begin{bmatrix} 4 & -3 \\ -2 & 5 \end{bmatrix}$$

$$= \frac{1}{14}\begin{bmatrix} 20(-2)^n - 6\cdot 2^n & -15(-2)^n + 15\cdot 2^n \\ 8(-2)^n - 8\cdot 2^n & -6(-2)^n + 20\cdot 2^n \end{bmatrix}$$

<u>Ex</u>: Find the intersection of
$3x + 2y - 4z = 6$ and $x - 3y - 2z = 4$

Solve $\begin{cases} x - 3y - 2z = 4 \\ 3x + 2y - 4z = 6 \end{cases}$

$$\begin{array}{cccc} x & y & z & \\ \multicolumn{4}{c}{\left[\begin{array}{ccc|c} 1 & -3 & -2 & 4 \\ 3 & 2 & -4 & 6 \end{array}\right]} \end{array}$$

$\vdots$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{-16}{11} & \frac{26}{11} \\ 0 & 1 & \frac{2}{11} & \frac{-6}{11} \end{array}\right] \quad \text{RREF}$$

$\vdots$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 26/11 \\ -6/11 \end{bmatrix} + \begin{bmatrix} 16/11 \\ -2/11 \end{bmatrix} t$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 26/11 \\ -6/11 \\ 0 \end{bmatrix} + \begin{bmatrix} 16/11 \\ -2/11 \\ 1 \end{bmatrix} t$$