251DX01Mar31

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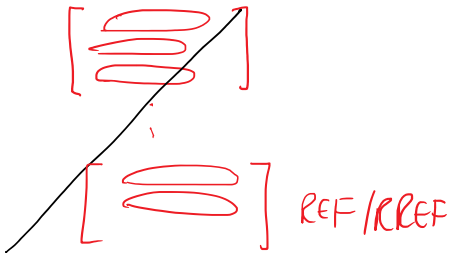
Ex: Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -3 \\ -6 \\ -21 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 9 \end{bmatrix}$

Find a basis for $\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$, where the basis vectors must be chosen from $\vec{v}_1, \vec{v}_2, \vec{v}_3$.



$$A = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 0 \\ 7 & -21 & 9 \end{bmatrix}$$

$R_2 - 2R_1$, $R_3 - 7R_1$

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

$R_3 + R_2$

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$ REF

Columns 1 and 3 of A

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 9 \end{bmatrix} \right\}$$

Practice #13 $\text{row}(A) = \text{col}(A^T)$

$$A^T = \begin{bmatrix} 1 & & & \\ 2 & & & \\ -1 & & & \\ 9 & & & \end{bmatrix}$$

Cols 1, 2, 4 of A^T $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 9 \end{bmatrix}, \dots \right\}$

= Rows 1, 2, 4 of A $\{ [1 \ 2 \ -1 \ 9] \dots \}$

Ex: T first rotates a vector by 70° (CCW) then doubles its x-component.

Find the image of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ under T.

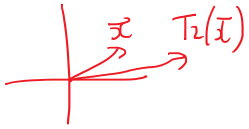
$\cos 70°$ $\sin 70°$

Find the image of $\begin{bmatrix}1\\1\end{bmatrix}$ under T.

$$[T_1] = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}_{\theta=70^\circ} = \begin{bmatrix}\cos 70^\circ & -\sin 70^\circ\\ \sin 70^\circ & \cos 70^\circ\end{bmatrix}$$

$$[T_2] = \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$$

$T_2\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ $T_2\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$



$$[T] = [T_2][T_1]$$

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = [T_2][T_1]\begin{bmatrix}1\\1\end{bmatrix}$$ order!

$$= \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\cos 70^\circ & -\sin 70^\circ\\ \sin 70^\circ & \cos 70^\circ\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

$$= \begin{bmatrix}2\cos 70^\circ & -2\sin 70^\circ\\ \sin 70^\circ & \cos 70^\circ\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

$$= \begin{bmatrix}2\cos 70^\circ - 2\sin 70^\circ\\ \sin 70^\circ + \cos 70^\circ\end{bmatrix}$$

$$\approx \begin{bmatrix}-1.2\\1.3\end{bmatrix}$$

Ex: a) Is $\left\{\begin{bmatrix}1\\0\\4\end{bmatrix}, \begin{bmatrix}-4\\0\\1\end{bmatrix}, \begin{bmatrix}0\\7\\0\end{bmatrix}\right\}$ an orthogonal basis for $\mathbb{R}^3$?

$\vec{v}_1\cdot\vec{v}_2 = 0$, $\vec{v}_1\cdot\vec{v}_3 = 0$, $\vec{v}_2\cdot\vec{v}_3 = 0$

vectors are orthogonal

YES n nonzero orthogonal vectors $\Rightarrow$ basis for $\mathbb{R}^n$

b) How many solutions does $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \begin{bmatrix}1\\2\\3\end{bmatrix}$ have?
($\vec{v}_1, \vec{v}_2, \vec{v}_3$ are basis vectors vectors above)

Let $\vec{w} = \begin{bmatrix}1\\2\\3\end{bmatrix}$

$$\vec{w} = \text{proj}_{\vec{v}_1}\vec{w} + \text{proj}_{\vec{v}_2}\vec{w} + \text{proj}_{\vec{v}_3}\vec{w}$$

(because basis is orthogonal)

$$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$$

ONE

Ex: Find A^{-1} using the adjoint method

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 3 & -6 & 1 \\ 2 & -2 & 2 \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$|A| = 1\begin{vmatrix} -6 & 1 \\ -2 & 2 \end{vmatrix} - 7\begin{vmatrix} \cdots \end{vmatrix} + 2\begin{vmatrix} \cdots \end{vmatrix}$$

$$= 1(-10) - 7(4) + 2(6)$$

$$= -26$$

Cofactor : signed determinant $\begin{bmatrix} + & - & + \\ - & + & - \\ \cdot & \cdot & \cdot \end{bmatrix}$

$$C = \begin{bmatrix} -10 & -4 & 6 \\ -18 & -2 & 16 \\ 19 & 5 & -27 \end{bmatrix}$$

$\uparrow$ $= \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$

$$A^{-1} = \frac{1}{|A|} C^T$$

$$= \frac{-1}{26} \begin{bmatrix} -10 & -18 & 19 \\ -4 & -2 & 5 \\ 6 & 16 & -27 \end{bmatrix}$$

Ex: $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ diagonalizes A to give $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

a) Find and simplify A^n.

$$A^n = PD^nP^{-1}$$

$$D^n = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$D^n = \begin{bmatrix} 2^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{bmatrix}$$

$$[P \mid I]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$\frac{R_3}{2}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right]$$

$R_1 - R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right]$$

P^{-1}

$$A^n = PD^nP^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2^n & 0 & 4^n \\ 0 & 3^n & 0 \\ 0 & 0 & 2(4^n) \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2^n & 0 & \frac{1}{2}(4^n - 2^n) \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{bmatrix}$$

$-\frac{1}{2}(2^n) + \frac{1}{2} \cdot 4^n$

$\frac{1}{2} \cdot 2 \cdot 4^n$

b) Find A^7

Sub $n = 7$

$$A^7 = \begin{bmatrix} 128 & 0 & 8128 \\ 0 & 2187 & 0 \\ 0 & 0 & 16384 \end{bmatrix}$$

Ex: Find all eigenvalues of $\begin{bmatrix} 10 & -2 \\ -3 & 11 \end{bmatrix}$

$$\det(A-\lambda I) = 0$$

$$\begin{vmatrix} 10-\lambda & -2 \\ -3 & 11-\lambda \end{vmatrix} = 0$$

$$(10-\lambda)(11-\lambda) - (-2)(-3) = 0$$

$$\lambda^2 - 21\lambda + \underbrace{110-6}_{104} = 0$$

$$\lambda = \frac{21 \pm \sqrt{(-21)^2 - 4\cdot 1\cdot 104}}{2}$$

$$\lambda = 13, 8$$

Ex: $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Find a basis for E_4

Solve $[A-4I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 \quad x_2 \quad x_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ RREF

$x_3 = t$

$x_1 - \frac{1}{2}x_3 = 0 \rightarrow x_1 = \frac{t}{2}$

$x_2 = 0$

$\vec{x} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} t \qquad (\vec{x} \neq \vec{0})$

$\text{Basis} = \left\{ \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$

Quick Ex: $(\lambda - 6)^2(\lambda - 7) = 0$

	Alg. Mult.	Geo. Mult.
$\lambda = 6$	2	1 or 2
$\lambda = 7$	1	1

basis vectors in E_λ

If G.M. of $\lambda = 6$ is 2, A is diagonalizable
" 1 " is Not "