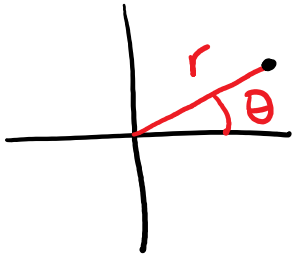
Week 6 Wednesday

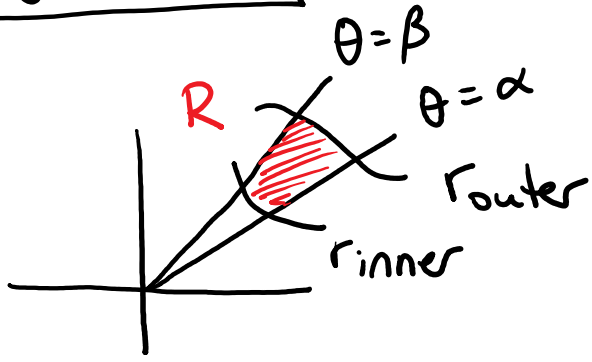
October 9, 2019 8:45 AM

13.4 Double Integration in Polar Coordinates

Recall polar coordinates



$$x = r\cos\theta$$
$$y = r\sin\theta$$



$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{r_{inner}}^{r_{outer}} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

r: correction factor

$dA = dx\,dy$ or $dy\,dx$

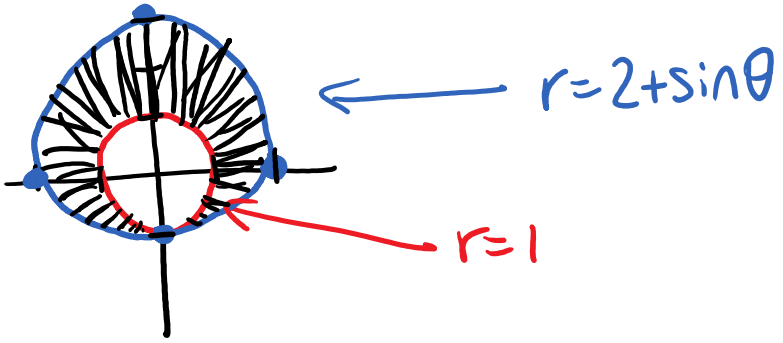
In polar coordinates:

$$dA = r\,dr\,d\theta$$

$$A = \iint r\,dr\,d\theta$$

$$V = \iint z\,r\,dr\,d\theta$$

Ex: Find the area inside $r = 2+\sin\theta$ and outside $r = 1$



R: $1 \le r \le 2+\sin\theta$

$0 \le \theta \le 2\pi$

$$A = \iint_R r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_1^{2+\sin\theta} r\,dr\,d\theta$$

$$A = \iint_R r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_1^{2+\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2}r^2\right]_{r=1}^{r=2+\sin\theta} d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2}(2+\sin\theta)^2 - \frac{1}{2}\right] d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2}(4+4\sin\theta+\sin^2\theta) - \frac{1}{2}\right] d\theta$$

$$\int_0^{2\pi} \left[2+2\sin\theta+\frac{\sin^2\theta}{2} - \frac{1}{2}\right] d\theta$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

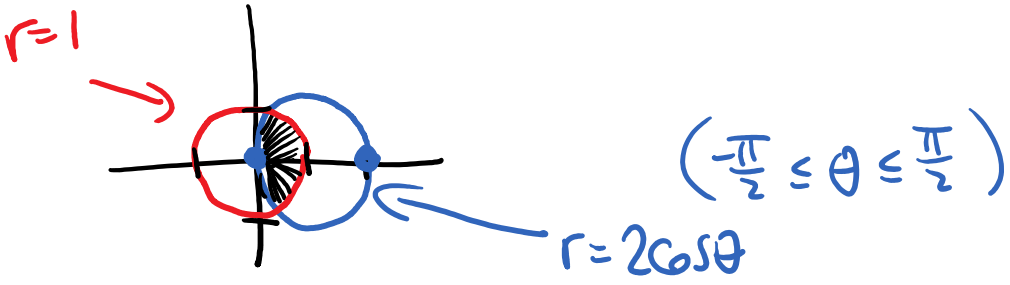
$$= \int_0^{2\pi} \left(\frac{3}{2} + 2\sin\theta + \frac{1-\cos 2\theta}{4}\right) d\theta$$

$$= \left[\frac{3}{2}\theta - 2\cos\theta + \frac{\theta}{4} - \frac{\sin 2\theta}{8}\right]_0^{2\pi}$$

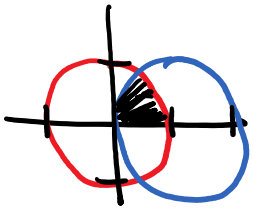
$$= \left(3\pi - 2 + \frac{\pi}{2}\right) - \left(-2\right)$$

$$= \frac{7\pi}{2}$$

Ex: Find area inside both circles $r=1$ and $r=2\cos\theta$



The area is double the following area:



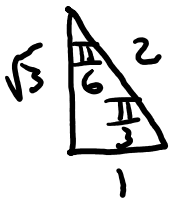
Intersection:

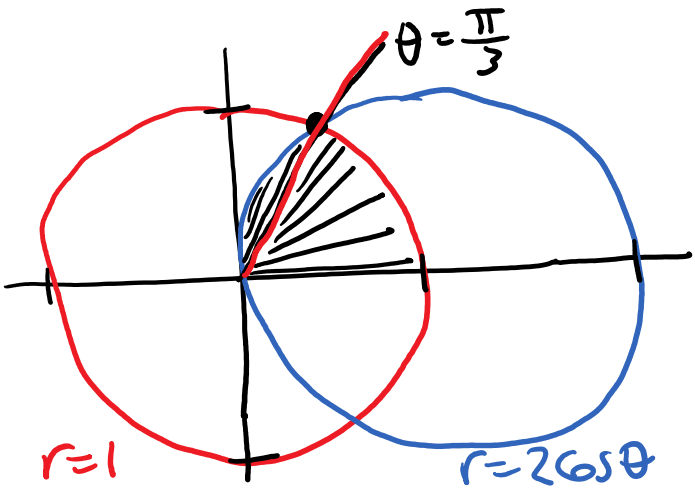
$r = r$

$1 = 2\cos\theta$

$\frac{1}{2} = \cos\theta$

$\theta = \frac{\pi}{3}, -\frac{\pi}{3}$





Area 1:
$0 \le r \le 1$
$0 \le \theta \le \frac{\pi}{3}$

Area 2:
$0 \le r \le 2\cos\theta$
$\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$

$$A = \iint r\,dr\,d\theta$$

$$= 2\left[\int_0^{\pi/3}\int_0^{1} r\,dr\,d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} r\,dr\,d\theta\right]$$

$$= 2\left[\int_0^{\pi/3}\left[\tfrac{1}{2}r^2\right]_0^{1} d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=2\cos\theta} d\theta\right]$$

$$= 2\left[\int_0^{\pi/3} \tfrac{1}{2}\,d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \tfrac{1}{2}(2\cos\theta)^2\,d\theta\right]$$

$2\cos^2\theta = 1+\cos 2\theta$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= 2\left[\left[\tfrac{1}{2}\theta\right]_{\theta=0}^{\theta=\frac{\pi}{3}} + \left[\theta + \frac{\sin 2\theta}{2}\right]_{\theta=\frac{\pi}{3}}^{\theta=\frac{\pi}{2}}\right]$$

$$= 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\theta=0}^{\theta=\frac{\pi}{3}}$$

$$= 2\left[\frac{\pi}{6} + \left(\frac{\pi}{2}\right) - \left(\frac{\pi}{3} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)\right]$$

$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

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$$= 2\left(\frac{\pi}{6} + \frac{\pi}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$$

$$= 2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$$

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$