1. [3 marks] Let $f(v,w) = e^{vw}$.
Find all the second-order partial derivatives of f.

$f_v = we^{vw}$ $\quad$ $f_w = ve^{vw}$

$f_{vv} = w^2e^{vw}$ $\quad$ $f_{vw} = e^{vw} + vwe^{vw}$

$f_{wv} = f_{vw}$ $\quad$ $f_{ww} = v^2e^{vw}$

2. [6 marks] Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$:
$xy^4 - 9x^3z + z^6 = y^2$

Take $\frac{\partial}{\partial x}$:

$$y^4 - 27x^2z - 9x^3\frac{\partial z}{\partial x} + 6z^5\frac{\partial z}{\partial x} = 0$$

$$[-9x^3 + 6z^5]\frac{\partial z}{\partial x} = -y^4 + 27x^2z$$

$$\frac{\partial z}{\partial x} = \frac{-y^4 + 27x^2z}{-9x^3 + 6z^5}$$

Take $\frac{\partial}{\partial y}$:

$$4xy^3 - 9x^3\frac{\partial z}{\partial y} + 6z^5\frac{\partial z}{\partial y} = 2y$$

$$[-9x^3 + 6z^5]\frac{\partial z}{\partial y} = 2y - 4xy^3$$

$$\frac{\partial z}{\partial y} = \frac{2y - 4xy^3}{-9x^3 + 6z^5}$$

3. [6 marks]
Find the equation of the tangent plane at the point where $x = 0$ and $y = 0$:
$z = 2xe^{2x} + 6\cos(2x + \frac{\pi}{4})\sin(2y + \frac{\pi}{4}) - \ln(7y + 1)$

2

$$z_x = 2e^{2x} + 4xe^{2x} - 12\sin(2x + \tfrac{\pi}{4})\sin(2y + \tfrac{\pi}{4})$$

$$z_y = 12\cos(2x + \tfrac{\pi}{4})\cos(2y + \tfrac{\pi}{4}) - \frac{7}{7y+1}$$

At $(0,0)$:

1

$$z_x = 2 - 12\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = -4$$

$$z_y = 12\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} - 7 = -1$$

1

$$\vec{n} = [-z_x, -z_y, 1] = [4, 1, 1]$$

1

At $(0,0)$ $z = 0 + 6\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} - 0 = 3$

$$\vec{n}\cdot\vec{x} = \vec{n}\cdot\vec{p}$$

1

$$\begin{bmatrix}4\\1\\1\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}4\\1\\1\end{bmatrix}\cdot\begin{bmatrix}0\\0\\3\end{bmatrix}$$

$$4x + y + z = 3$$

4. [5 marks] Use differentials to estimate the change in f from $(x, y) = (4, 8)$ to $(x, y) = (4 + h, 7.97)$. Your answer will involve h.

$f = x^{\frac{1}{2}} y^{\frac{1}{3}}$

$$f_x = \frac{1}{2} x^{-1/2} y^{1/3}$$
$$f_y = \frac{1}{3} x^{1/2} y^{-2/3}$$

(2)

At $(4,8)$: $f_x = \frac{1}{2} \frac{1}{\sqrt{4}} \sqrt[3]{8} = \frac{1}{2}$

$f_y = \frac{1}{3} \sqrt{4} \frac{1}{\sqrt[3]{8}^{\,2}} = \frac{1}{6}$

Sub $x=4$ $y=8$ $dx = h$ $dy = -0.03$ (1)

$$\Delta f \approx df$$
$$\approx f_x dx + f_y dy$$
$$\approx \frac{1}{2} h + \frac{1}{6}(-0.03)$$
$$\approx 0.5h - 0.005$$

(2)

5. [6 marks] Find all the critical points (x, y) of the following function:
$f = 25xy - 3x^2y - 2xy^2$

$f_x = 25y - 6xy - 2y^2$

$25y - 6xy - 2y^2 = 0$

$y(25 - 6x - 2y) = 0$ ①

$f_y = 25x - 3x^2 - 4xy$

$25x - 3x^2 - 4xy = 0$

$x(25 - 3x - 4y) = 0$ ②

There are 4 cases.

Case 1 $y = 0$ and $x = 0$ $\Rightarrow (0, 0)$

Case 2 $y = 0$ and $25 - 3x - 4y = 0$

$25 - 3x = 0 \Rightarrow (\frac{25}{3}, 0)$

Case 3 $x = 0$ and $25 - 6x - 2y = 0$

$25 - 2y = 0 \Rightarrow (0, \frac{25}{2})$

Case 4: $\begin{cases} 25 - 6x - 2y = 0 \\ 25 - 3x - 4y = 0 \end{cases} \Rightarrow \begin{matrix} 6x + 2y = 25 & ③ \\ 3x + 4y = 25 & ④ \end{matrix}$

③ $- 2 \times$ ④: $-6y = -25$

$y = \frac{25}{6} \to$ ③: $6x + \frac{25}{3} = 25$

$x = \frac{25}{9}$

$\Rightarrow (\frac{25}{9}, \frac{25}{6})$

1 mark for: f_x, f_y, each case