Math 250B

Practice Questions for Math 250B

1. Identify the surfaces of equation

   (a) $x^2 + y^2 + z^2 = 6z$

   (b) $x^2 + 2y^2 + z^2 + 4y + 3 = 4x + 2z$.

   Answers: (a) A sphere centered at $(0, 0, 3)$ of radius 3. (b) An ellipsoid centered at $(2, -1, 1)$.

2. Evaluate the following limits or show that it does not exist.

   (a) $\displaystyle\lim_{(x,y)\to(1,2)} \frac{xy}{x^2 + y^2}$ (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$

   Answers: (a) $2/5$ (b) The limit does not exist.

3. Describe the domain of the following function.

   $$f(x, y) = \frac{\sqrt{x}\,\ln(y - 2x)}{\sqrt{4 - x^2 - y^2}}$$

   Answer: Domain $= \big\{(x, y) \mid x \geq 0,\ y > 2x,\ x^2 + y^2 < 4\big\}$

4. Consider the function

   $$f(x, y, z) = x^2 e^{-2y} z^3 + \sin(x^2 y).$$

   Evaluate (a) $f_y(3, 0, 2)$ (b) $f_{xz}(2, -1, 1)$

   Answers: (a) -135 (b) $12e^2 \approx 88.67$

5. The two-dimensional Laplace's equation is

   $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

   Show that

   $$z = xy + e^{-y} \sin x$$

   satisfies Laplace's equation.

6. The centripetal acceleration of an object moving in a circle is $a = \frac{v^2}{r}$ where v is the velocity and r is the radius of the circle. Use differentials to estimate the absolute error of the centripetal acceleration if

   $$v = (10.0 \pm 0.2)\ \text{m/s} \qquad \text{and} \qquad r = (4.0 \pm 0.1)\ \text{m}.$$

   Answer: $\Delta a \approx \pm 1.63\ \text{m/s}^2$

7. Use differentials to estimate the value of $f(3.1, 1.8)$ given that $f(3, 2) = 5$ and $\nabla f(3, 2) = 4\mathbf{i} - \mathbf{j}$.

   Answer: $f(3.1, 1.8) \approx 5.6$

8. For an open box with constant volume 8 m^3, the total surface area A of the sides and the base is a function of the width w and the height h (both in meters)

$$A = 2hw + \frac{16}{w} + \frac{8}{h}.$$

Use differentials to estimate the maximum relative error in the surface area if

$$w = 1.0 \pm 0.1 \text{ m} \qquad \text{and} \qquad h = 4.0 \pm 0.1 \text{ m}.$$

Answer: $\left(\frac{\Delta A}{A}\right)_{\max} \approx 3.65\,\%$

9. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^2y + xz + z^3 = 0$.

Answers: $\frac{\partial z}{\partial x} = \frac{-(2xy+z)}{x+3z^2}$ and $\frac{\partial z}{\partial y} = \frac{-x^2}{x+3z^2}$

10. A rectangular box has a square base. Find the rate of change of the volume (in m^3/min) if its base edges are increasing at 2 cm/min and its height is decreasing at 3 cm/min at the instant when each dimension is 1 meter.

Answer: 0.01 m^3/min

11. Use the chain rule to find both $\dfrac{\partial w}{\partial s}$ and $\dfrac{\partial w}{\partial t}$ evaluated at $s = 2$, $t = 1$ if

$$w = x^3 - 3x^2y + xz^2$$

where

$$x = s^2 + t^2, \qquad y = s^2 - t^2, \qquad z = s + t.$$

Answers: $\dfrac{\partial w}{\partial s} = -294$ and $\dfrac{\partial w}{\partial t} = 168$

12. Suppose the function

$$T = 10 + \frac{xy}{3} + \frac{xz}{6} + \frac{yz}{6}$$

gives the temperature at point (x, y, z) in space. (The units of T is $^\circ C$ and the units of x, y, z are km.) What time rate of change (in $^\circ C$/h) will an eagle observe as it flies through point $P(1, 1, 1)$ at a speed of 20 km/h heading directly toward the point $Q(2, 3, 3)$?

Answer: 14.4 $^\circ C$/h

13. The voltage V in a simple electrical circuit is slowly decreasing as the battery wears out. The resistance R is slowly increasing as the resistor heats up. Use Ohm's law, $V = RI$, to compute how the current I is changing at the moment when $R = 400\,\Omega$, $I = 0.08$ A, $\frac{dV}{dt} = -0.01$ V/s, and $\frac{dR}{dt} = 0.03\ \Omega$/s.

Answer: -3.1×10^{-5} A/s $= -31\,\mu$A/s

14. Find Find $\nabla f(x_0, y_0)$ given that

$$D_{\mathbf{u}} f(x_0, y_0) = -1, \quad \text{for } \mathbf{u} = -\tfrac{4}{5}\,\mathbf{i} + \tfrac{3}{5}\,\mathbf{j}$$

and

$$D_{\mathbf{v}} f(x_0, y_0) = 2, \quad \text{for } \mathbf{v} = \tfrac{3}{5}\,\mathbf{i} + \tfrac{4}{5}\,\mathbf{j}.$$

Answer: $\nabla f(x_0, y_0) = 2\mathbf{i} + \mathbf{j} = \langle 2, 1 \rangle$.

15. The temperature in Celcius on a metal plate is given by $T = 100 - x^2 - 2y^2$.
   (a) Find the directional derivative of T at point $(4, 3)$ in the direction of $\mathbf{v} = \langle 2, 1\rangle$.
   (b) Find a unit vector pointing in the direction of maximum increase of T at point $(4, 3)$.

   Answers: (a) $-28/\sqrt{5}$ (b) $\frac{-1}{\sqrt{13}}\langle 2, 3\rangle$

16. Find an equation of the tangent plane to the surface of equation $z = 3x^2 + y^2$ at point $(1, 2, 7)$.

   Answer: $6x + 4y - z = 7$

17. (4 points) Find parametric equations for the normal line to the ellipsoid of equation

$$2x^2 + y^2 + 3z^2 = 9$$

   at point $(1, 2, -1)$.

   Answer: $\begin{cases} x = 1 + 4t \\ y = 2 + 4t \\ z = -1 - 6t \end{cases}$

18. Find a vector tangent to the curve of intersection of the surfaces

$$x^2 + y^2 - z = 0 \qquad \text{and} \qquad xyz = 10$$

   at point $(1, 2, 5)$.

   Answer: $\langle 13, -14, -30\rangle$

19. The surface

$$z = \frac{8}{3}x^3 + 4y^3 - x^4 - y^4$$

   opens downward and has a highest point. Find the coordinate (x, y, z) of the highest point on the surface.

   Answer: $(2, 3, 97/3)$

20. Find the point on the plane $2x + y - z = 8$ that is closest to the point $(2, 1, 0)$.

   Answer: $(3, 3/2, -1/2)$

21. Find the dimensions that minimize the total cost of a rectangular box with a volume of 6 m^3 if the front and back cost $\$4/\text{m}^2$, the top and bottom cost $\$6/\text{m}^2$, and the two other sides cost $\$2/\text{m}^2$

   Answer: The dimensions that minimize the cost are $1\,\text{m} \times 2\,\text{m} \times 3\,\text{m}$ for a cost of 72 dollars.

22. Find the absolute maximum and minimum values of $f(x, y) = 3x^2 + 2y^2 - 4y$ on the region in the xy-plane bounded by the graphs of $y = x^2$ and $y = 4$ At what points are these extremum values attained?

   Answers: Absolute maximum value is 28, attained at $(\pm 2, 4)$. Absolute minimum value is -2, attained at $(0, 1)$.

23. Find all critical points of $f(x, y) = x^3 + xy^2 - 3x^2 - 4y^2 + 4$ and test if they correspond to relative minimum, relative maximum, or saddle point.

   Answers: $(0, 0) \to$ relative max, $(2, 0) \to$ saddle point.

24. Find all critical points of

$$f(x,y) = x^3 + y^2 - 3xy + 1$$

and test if they correspond to relative minimum, relative maximum, or saddle point.

Answers: $(x,y) = (0,0) \to$ a saddle point, $(x,y) = (3/2, 9/4) \to$ relative min.

25. Use Lagrange multipliers to find the points on the curve $x^2y = 16$ that are closest to the origin.

Answers: The closest points to the origin are $(\pm 2\sqrt{2}, 2)$.

26. Find the minimum value of $f(x,y,z) = 3x + 2y + z$ on the paraboloid $z = 9x^2 + 4y^2$. At what point is the minimum value attained? What about the maximum value?

Answers: The minimum value is $-\frac{1}{2}$ and is attained at $(x,y,z) = (-\frac{1}{6}, -\frac{1}{4}, \frac{1}{2})$. The function does not have a maximum value on the paraboloid.

27. The curve $17x^2 + 12xy + 8y^2 = 100$ is an ellipse centered at the origin. Use the method of Lagrange multipliers to locate the points on the ellipse that are closest and farthest from the origin.

Answers: Closest points: $(-2,-1)$ and $(2,1)$. Farthest points: $(2,-4)$ and $(-2,4)$.

28. Use the method of Lagrange multipliers to find the points on the surface $z = xy + 5$ closest to the origin.

Answers: $(2,-2,1)$ and $(-2,2,1)$.

29. Evaluate $\iint_R (x+y)\,dA$ where R is the region bounded by $y = \sqrt{x}$ and $y = x^2$.

Answer: 3/10

30. Locate the center of mass of the first-quadrant region bounded by $y = x^2$, $x = 0$, and $y = 1$ with density $\delta(x,y) = xy$.

Answer: $(\overline{x}, \overline{y}) = \left(\frac{4}{7}, \frac{3}{4}\right)$

31. Evaluate the following double integrals.

(a) $I = \displaystyle\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\,dy\,dx$ (b) $I = \displaystyle\int_0^2 \int_{y/2}^1 \sin(x^2)\,dx\,dy$ (c) $I = \displaystyle\int_0^{\ln 2} \int_{e^x}^2 \frac{1}{\ln y}\,dy\,dx$.

Answers: (a) $(e^8 - 1)/4$, (b) $1 - \cos(1)$, (c) 1.

32. Evaluate $\iint_R y^2\,dA$ where R is the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Answer: $\frac{15\pi}{4}$

33. Evaluate

$$\int_0^3 \int_0^{\sqrt{9-x^2}} (x^2 + y^2)^{3/2}\,dy\,dx$$

Answer: $\frac{243\pi}{10}$

34. Set-up a double integral in polar coordinates to evaluate the following.

(a) $\iint_R \sqrt{1 + x^2 + y^2}\,dA$ for the region R defined by

$$x \geq 0, \quad 0 \leq y \leq x, \quad x^2 + y^2 \leq 9.$$

(b) $\iint_R x\,dA$ for the region R defined by

$$1 \le x^2 + y^2 \le 2, \quad y \ge 0, \quad 0 \le x \le \sqrt{3}\,y.$$

(c) $\iint_R x^2\,dA$ where R is the first quadrant region outside the circle $r = 1$ and inside the cardioid $r = 1 + \cos\theta$.

(d) The volume of the part of the ball $x^2 + y^2 + z^2 \le 4$ that lies in the first octant and inside the cylinder $x^2 + y^2 = 2x$.

(e) The volume of the solid bounded by $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$.

Answers: (a) $\displaystyle\int_0^{\pi/4}\int_0^3 r\sqrt{1+r^2}\;dr\,d\theta$ (b) $\displaystyle\int_{\pi/6}^{\pi/2}\int_1^{\sqrt{2}} r^2\cos\theta\;dr\,d\theta$

(c) $\displaystyle\int_0^{\pi/2}\int_1^{1+\cos\theta} r^3\cos^2\theta\;dr\,d\theta$ (d) $\displaystyle V = \int_0^{\pi/2}\int_0^{2\cos\theta} r\sqrt{4-r^2}\;dr\,d\theta$

(e) $\displaystyle V = \int_0^{2\pi}\int_0^2 (8r - 2r^3)\;dr\,d\theta$

35. Evaluate

$$\int_0^2\int_0^x\int_0^{x+y} e^x(y+2z)\,dz\,dy\,dx$$

Answer: $19\left(\dfrac{e^2}{3}+1\right) \approx 65.8$

36. Set-up an iterated triple integral for $\iiint_Q f(x,y,z)\,dV$ for the following solid regions Q.

(a) Q is the first-octant region under the plane of equation $3x + 2y + z = 6$.

(b) Q is the region in the first octant bounded by the cylinder $z = 1 - y^2$ and lying between the planes $x + y = 1$ and $x + y = 3$.

(c) Q is the upper hemisphere given by $z = \sqrt{1 - x^2 - y^2}$.

(d) Q is the region bounded below by the paraboloid $z = x^2 + y^2$ and bounded above by the sphere $x^2 + y^2 + z^2 = 6$.

Answers:

(a) $\displaystyle\int_0^2\int_0^{(6-3x)/2}\int_0^{6-3x-2y} f(x,y,z)\;dz\,dy\,dx$ (b) $\displaystyle\int_0^1\int_{1-y}^{3-y}\int_0^{1-y^2} f(x,y,z)\;dz\,dx\,dy$

(c) $\displaystyle\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$ (d) $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}}\int_{x^2+y^2}^{\sqrt{6-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$

37. Find the surface area of the part of the paraboloid $z = 10 - x^2 - y^2$ that lies between the two planes $z = 1$ and $z = 6$.

Answer: $\frac{\pi}{6}\left(37^{3/2} - 17^{3/2}\right) \approx 81.14$

38. Set-up a triple integral in cylinder coordinates to compute $\iiint_Q (x^2 + y^2)\,dV$ where Q is the solid region in the first octant that is inside the hemisphere $z = \sqrt{9 - x^2 - y^2}$ and outside the cylinder $x^2 + y^2 = 2y$

Answer: $\displaystyle\int_0^{\pi/2}\int_{2\sin\theta}^3\int_0^{\sqrt{9-r^2}} r^3\;dz\,dr\,d\theta$

39. Set-up a triple integral in spherical coordinates to evaluate $\iiint_Q (x^2+y^2)\,dV$ where Q is the region that lies between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=9$, and inside the cone $z=\sqrt{4x^2+4y^2}$.

    Answer: $\displaystyle\int_0^{2\pi}\int_0^{\arctan(1/2)}\int_1^3 \rho^4\sin^3\phi\;d\rho\,d\phi\,d\theta$

40. Evaluate
$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} \sqrt{x^2+y^2+z^2}\;dz\,dy\,dx.$$

    Answer: $\pi(2-\sqrt{2})$

41. Use spherical coordinates to find the mass of the solid region above the cone $z=\sqrt{x^2+y^2}$ and below $z=3$ with density $\delta(x,y,z)=\sqrt{x^2+y^2+z^2}$.

    Answer: $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{3\sec\phi}\rho^3\sin\phi\;d\rho\,d\phi\,d\theta=\frac{27\pi}{2}(2\sqrt{2}-1)$

42. Evaluate $\iiint_B (x^2+y^2)\,dV$ where $B=\{(x,y,z)\,|\,x^2+y^2+z^2\le 1\}$ is the unit ball in $\mathbb{R}^3$.

    Answer: $8\pi/15$

43. Let R be the parallelogram region with vertices at $(0,0)$, $(3,-3)$, $(5,-2)$, and $(2,1)$. Evaluate the double integral $\iint_R (x+y)\,dA$ by using an appropriate change of variables.

    Answer: By using $u=x+y$ and $v=x-2y$, we get $\iint_R (x+y)\,dA = 27/2$.

    Note that if we used (x,y) coordinates, we would need to evaluate three double integrals.
$$\int_0^2\int_{-x}^{x/2}(x+y)\,dy\,dx+\int_2^3\int_{-x}^{3-x}(x+y)\,dy\,dx+\int_3^5\int_{(x-9)/2}^{3-x}(x+y)\,dy\,dx=\frac{27}{2}$$

44. Let R be the region in the first quadrant bounded by the curves $xy=1$, $xy=2$, $y=x$, and $y=3x$. Evaluate $\iint_R y^2\,dA$ by using the change of variables $u=xy$ and $v=y/x$.

    Answer: $3/2$

45. Find the divergence and curl of $\mathbf{F}=x^2yz\,\mathbf{i}+3xyz^3\,\mathbf{j}+(x^2-z^2)\,\mathbf{k}$.

    Answers:
$$\nabla\cdot\mathbf{F}=2xyz+3xz^3-2z$$
$$\nabla\times\mathbf{F}=-9xyz^2\,\mathbf{i}+(x^2y-2x)\,\mathbf{j}+(3yz^3-x^2z)\,\mathbf{k}$$

46. Determine if $\mathbf{F}=(e^x\cos y+yz)\,\mathbf{i}+(xz-e^x\sin y)\,\mathbf{j}+xy\,\mathbf{k}$ is conservative and if yes, find a potential function.

    Answer: The field is conservative and $f=e^x\cos y+xyz+C$ is a potential function for any constant C, i.e., $\mathbf{F}=\nabla f$.

47. Evaluate the line integral $\int_C z\,ds$ for the curve C with parametrization
$$x=3\cos t,\quad y=3\sin t,\quad z=4t,\quad 0\le t\le\pi.$$

    Answer: $10\pi^2$

48. Evaluate the line integral $\int_C (x+3y+2z)\,ds$ where C is the straight line segment from $(1,-1,2)$ to $(2,1,5)$.

    Answer: $\frac{17\sqrt{14}}{2}$

49. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = xy\,\mathbf{i} + x^2\,\mathbf{j}$ and the path C is described by

$$\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j}, \quad 0 \le t \le 1.$$

    Answer: 5/7

50. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = 4x^3\,\mathbf{i} + 2yz^3\,\mathbf{j} + 3y^2z^2\,\mathbf{k}$ and the path C is described by

$$\mathbf{r}(t) = \sqrt{t^3+1}\,\mathbf{i} + (t^2-1)\,\mathbf{j} + (t+1)\,\mathbf{k}, \quad 0 \le t \le 2.$$

    Answer: Use the fundamental theorem of line integrals to get $\int_C \mathbf{F}\cdot d\mathbf{r} = 322$.

51. Evaluate

$$\oint_C (2x+y^2)\,dx + (x^2+2y)\,dy$$

    where C is the positively oriented closed curve formed by $y=0$, $x=2$, and $y=x^3/4$.

    Answer: 72/35

52. Evaluate

$$\oint_C \left(2y + \sqrt{3+x^3}\right)\,dx + \left(5x + e^{y^5}\right)\,dy$$

    where C is the positively oriented circle $x^2+y^2=9$.

    Answer: 27π

53. Find the surface area of the surface described by

$$\mathbf{r}(u,v) = a\sin u\cos v\,\mathbf{i} + a\sin u\sin v\,\mathbf{j} + a\cos u\,\mathbf{k}, \quad 0 \le u \le \pi,\ 0 \le v \le 2\pi.$$

    Answer: $4\pi a^2$

54. Evaluate the surface integral $\iint_S z\,dS$ where S is the portion of the cone $z^2 = x^2+y^2$ from $z=0$ to $z=1$.

    Answer: $\dfrac{2\sqrt{2}\pi}{3}$

55. Evaluate the outward flux of the vector field

$$\mathbf{F}(x,y,z) = x^3\,\mathbf{i} + 2z^2\,\mathbf{j} + 3zy^2\,\mathbf{k}$$

    across the closed surface S bounded by the paraboloid $z = 2x^2+2y^2$ and the plane $z=8$.

    Answer: $\Phi = 64\pi$