(21) a) This is a p-series with $p = 1.1$
The series converges.

b) Let $f(x) = x^2 e^{-x^3}$
$f(x)$ is continuous on $[1, \infty)$ ✓
$f(x) > 0$ on $[1, \infty)$ ✓

$$f'(x) = x^2\left[-3x^2 e^{-x^3}\right] + 2x e^{-x^3}$$
$$= (-3x^4 + 2x)e^{-x^3}$$

$f'(x) < 0$ on $[1, \infty)$
$f(x)$ is decreasing on $[1, \infty)$ ✓

$$\int_1^{\infty} x^2 e^{-x^3}\,dx = -\frac{1}{3}\int_{-1}^{-\infty} e^u\,du$$

$u = -x^3$
$du = -3x^2\,dx$
$-\frac{1}{3}du = x^2\,dx$
$x = 1 \Rightarrow u = -1$
$x \to \infty \Rightarrow u \to -\infty$

$$= \frac{1}{3}\int_{-\infty}^{-1} e^u\,du$$

$$= \lim_{a \to -\infty} \frac{1}{3}\int_a^{-1} e^u\,du$$

$$= \lim_{a \to -\infty} \frac{1}{3} e^u \Big|_a^{-1}$$

$$= \lim_{a \to -\infty} \frac{1}{3}e^{-1} - \frac{1}{3}e^a$$

$$= \frac{1}{3}e^{-1}$$

The series converges by the Integral Test.

c) $\lim_{n \to \infty} \frac{n}{2n+1} \neq 0$

The series diverges by the n^{th} term test.

d) Alternating Series ✓

$a_n = \frac{1}{2n}$

$\lim_{n \to \infty} a_n = 0$ ✓

$\frac{1}{2(n+1)} \leq \frac{1}{2n}$ for $n \geq 1$ ✓

The series converges by the Alternating Series Test.

e) Let $a_n = \frac{1}{3^{(n^3)}}$ $\quad b_n = \frac{1}{3^n}$

$0 < a_n \leq b_n$ for $n \geq 1$ ✓

$\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges (geometric)

$\Rightarrow \sum_{n=1}^{\infty} a_n$ converges by the Direct Comparison Test.

f) Analyzing the dominant terms (and ignoring coefficients), a term looks like $\frac{1}{n^{3/2}}$.

Let $a_n = \frac{\sqrt{n}+3}{7n^2+2}$ and $b_n = \frac{1}{n^{3/2}}$.

$a_n, b_n > 0$ ✓

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\sqrt{n}+3}{7n^2+2}\right) n^{3/2}$$

$$= \lim_{n\to\infty} \frac{n^2+3n^{3/2}}{7n^2+2}$$

$$= \lim_{n\to\infty} \frac{2n+\frac{9}{2}n^{1/2}}{14n} \quad \text{(L'Hôpital)}$$

$$= \lim_{n\to\infty} \frac{2+\frac{9}{4}n^{-1/2}}{14} \quad \text{(L'Hôpital)}$$

$$= \frac{1}{7}$$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (p-series)

$\Rightarrow \sum_{n=1}^{\infty} a_n$ converges by the Limit Comparison Test.

g) Let $a_n = \dfrac{n!}{1(3)(5)\cdots(2n-1)}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)!}{1(3)(5)\cdots(2n+1)} \cdot \frac{1(3)(5)\cdots(2n-1)}{n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+1}{2n+1} \right|$$

$$= \lim_{n\to\infty} \frac{n+1}{2n+1}$$

$$= \lim_{n\to\infty} \frac{1}{2} \quad \text{(L'Hôpital's Rule)}$$

$$= \frac{1}{2}$$

The series converges by the Ratio Test.

h) $$\lim_{n\to\infty} \sqrt[n]{|a_n|}$$

$$= \lim_{n\to\infty} \left| \frac{1}{1+\ln n} \right|$$

$$= \lim_{n\to\infty} \frac{1}{1+\ln n}$$

$$= 0$$

The series converges by the Root Test.

(22) $f(x) = e^x$

$f^{(n)}(x) = e^x$

$x = -0.2$

$c = 0$ (Maclaurin polynomial)

$$|R_N(x)| = \left| \frac{f^{(N+1)}(z)}{(N+1)!} (x-c)^{N+1} \right|$$

$$|R_N(-0.2)| = \left| \frac{e^z}{(N+1)!} (-0.2)^{N+1} \right|$$

where z is between -0.2 and 0

$$\boxed{\begin{array}{l} e^z \text{ is increasing on } -0.2 \le z \le 0 \\ \Rightarrow \text{ use } e^0 = 1 \end{array}}$$

$$\le \frac{1}{(N+1)!} (0.2)^{N+1}$$

N	$\frac{1}{(N+1)!}(0.2)^{N+1} < 0.0001$
1	No
2	No
3	YES

Therefore $N \geq 3$.

(23) $a_n = \dfrac{(x-1)^n}{n^2 \cdot 2^n}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{(n+1)^2 \cdot 2^{n+1}} \cdot \frac{n^2 \cdot 2^n}{(x-1)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x-1}{2} \cdot \frac{n^2}{(n+1)^2} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x-1}{2} \cdot \frac{2n}{2(n+1)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x-1}{2} \cdot \frac{2}{2} \right|$$

$$= \left| \frac{x-1}{2} \right|$$

We used L'Hôpital's Rule twice.

The series converges when $\left| \frac{x-1}{2} \right| < 1$

$$|x-1| < 2$$

$$-2 < x-1 < 2$$

$$-1 < x < 3$$

When $x = -1$, series $= \displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2 \cdot 2^n}$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n^2 \cdot 2^n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

Converges by the Alternating Series Test.

When $x = 3$, series $= \sum_{n=1}^{\infty} \frac{2^n}{n^2 \cdot 2^n}$

$= \sum_{n=1}^{\infty} \frac{1}{n^2}$

Converges (p-series with $p = 2$).

The Interval of convergence is $-1 \leq x \leq 3$.

(24) $$e^{x} \simeq 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$e^{-2x} \simeq 1 - 2x + \frac{4x^2}{2} - \frac{8x^3}{6}$$

$$\simeq 1 - 2x + 2x^2 - \frac{4x^3}{3}$$

$$xe^{-2x} \simeq x - 2x^2 + 2x^3 - \frac{4x^4}{3}$$

(25) $(1+x)^k \simeq 1 + kx + \frac{k(k-1)}{2} x^2$

$(1+x^2)^k \simeq 1 + kx^2 + \frac{k(k-1)}{2} x^4$

$(1+x^2)^{1/3} \simeq 1 + \frac{x^2}{3} + \frac{1}{2}\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right) x^4$

$\simeq 1 + \frac{x^2}{3} - \frac{x^4}{9}$

$\int_0^{0.5} \sqrt[3]{1+x^2}\, dx \simeq \int_0^{0.5} \left(1 + \frac{x^2}{3} - \frac{x^4}{9}\right) dx$

$\simeq \left[x + \frac{x^3}{9} - \frac{x^5}{45}\right]_0^{0.5}$

$\simeq 0.5132$