(21) a) This is a p-series with $p = 0.9$
The series diverges.

b) Let $f(x) = xe^{-x^2}$
$f(x)$ is continuous on $[1, \infty)$ ✓
$f(x) > 0$ on $[1, \infty)$ ✓

$$f'(x) = x\left[-2xe^{-x^2}\right] + e^{-x^2}$$
$$= \left(-2x^2+1\right)e^{-x^2}$$

$f'(x) < 0$ on $[1, \infty)$
$f(x)$ is decreasing on $[1, \infty)$ ✓

$$\int_1^{\infty} xe^{-x^2}\,dx = -\frac{1}{2}\int_{-1}^{-\infty} e^u\,du$$

$$\boxed{\begin{aligned} u &= -x^2 \\ du &= -2x\,dx \\ -\tfrac{1}{2}du &= x\,dx \\ x = 1 &\Rightarrow u = -1 \\ x \to \infty &\Rightarrow u \to -\infty \end{aligned}}$$

$$= \frac{1}{2}\int_{-\infty}^{-1} e^u\,du$$

$$= \lim_{a \to -\infty} \frac{1}{2}\int_a^{-1} e^u\,du$$

$$= \lim_{a \to -\infty} \frac{1}{2}e^u\Big|_a^{-1}$$

$$= \lim_{a \to -\infty} \frac{1}{2}e^{-1} - \frac{1}{2}e^a$$

$$= \frac{1}{2}e^{-1}$$

$\boxed{\text{Series Converges by the Integral Test.}}$

c) $\lim_{n\to\infty} \frac{n+1}{n+2} \neq 0$

The series diverges by the n^{th} term test.

d) Let $b_n = \frac{1}{n}$ and $a_n = \frac{1+|\cos n|}{n}$

$$0 < b_n \leq a_n \quad \text{for } n \geq 1.$$

$\sum_{n=1}^{\infty} b_n$ diverges (p-series)

$\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges by Direct Comparison.

e) Alternating series

$a_n = \frac{1}{\sqrt{n}}$

$\lim_{n\to\infty} a_n = 0$

$\frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}}$ for $n \geq 1$

The series converges by the Alternating Series Test.

f) $\lim_{n\to\infty} \sqrt[n]{|a_n|}$

$= \lim_{n\to\infty} \frac{3n+4}{4n+3}$

$= \lim_{n\to\infty} \frac{3}{4}$ by l'Hôpital's Rule

$= \frac{3}{4}$

The series converges by the Root Test.

g) Let $a_n = \frac{2^n}{n!}$

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$= \lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right|$

$= \lim_{n\to\infty} \left| \frac{2}{n+1} \right|$

$= 0$

The series converges by the Ratio Test.

h) Analyzing the dominant terms (and ignoring coefficients), a term looks like $\frac{1}{\sqrt{n}}$.

Let $a_n = \frac{\sqrt{n}+3}{7n+2}$ and $b_n = \frac{1}{\sqrt{n}}$

$a_n, b_n > 0$ ✓

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\sqrt{n}+3}{7n+2}\right)\sqrt{n}$$

$$= \lim_{n\to\infty} \frac{n + 3\sqrt{n}}{7n+2}$$

$$= \lim_{n\to\infty} \frac{1 + \frac{3}{2}n^{-1/2}}{7} \quad \text{(L'Hôpital)}$$

$$= \frac{1}{7}$$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series)

$\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges by the Limit Comparison Test.

(22) a) $f(x) = \cos x \qquad f\left(\frac{\pi}{2}\right) = 0$

$f'(x) = -\sin x \qquad f'\left(\frac{\pi}{2}\right) = -1$

$f''(x) = -\cos x \qquad f''\left(\frac{\pi}{2}\right) = 0$

$f'''(x) = \sin x \qquad f'''\left(\frac{\pi}{2}\right) = 1$

$f^{(4)}(x) = \cos x \qquad f^{(4)}\left(\frac{\pi}{2}\right) = 0$

$$P_4(x) = f\left(\tfrac{\pi}{2}\right) + f'\left(\tfrac{\pi}{2}\right)\left(x-\tfrac{\pi}{2}\right) + \frac{f''\left(\frac{\pi}{2}\right)}{2!}\left(x-\tfrac{\pi}{2}\right)^2 + \dots$$

$$+ \frac{f^{(4)}\left(\frac{\pi}{2}\right)}{4!}\left(x-\tfrac{\pi}{2}\right)^4$$

$$= -\left(x-\tfrac{\pi}{2}\right) + \frac{1}{3!}\left(x-\tfrac{\pi}{2}\right)^3$$

b) $P_4(1.5) = -\left(1.5-\frac{\pi}{2}\right) + \frac{1}{3!}\left(1.5-\frac{\pi}{2}\right)^3$

$\simeq 0.0707372186$

c) $$|R_n(x)| = \left|\frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}\right|$$

where z is between x and c

$f^{(5)}(x) = -\sin x$

$$|R_4(1.5)| = \left|\frac{-\sin z}{5!}\left(1.5-\tfrac{\pi}{2}\right)^5\right|$$

$\boxed{\sin z \text{ is increasing for } 1.5 \le z \le \frac{\pi}{2}}$

$$\leq \left| \frac{-\sin\frac{\pi}{2}}{5!} \left(1.5 - \frac{\pi}{2}\right)^5 \right|$$

$$\leq 0.000\,000\,02$$

d) $0.070737186 - 0.00000002 \leq \cos 1.5 \leq$
$0.070737186 + 0.00000002$

$$0.070737166 \leq \cos 1.5 \leq 0.070737206$$

(23) $a_n = \dfrac{(x-4)^n}{n \cdot 6^n}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-4)^{n+1}}{(n+1)\, 6^{n+1}} \cdot \frac{n \cdot 6^n}{(x-4)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x-4}{6} \cdot \frac{n}{n+1} \right|$$

$$= \left| \frac{x-4}{6} \right| \quad \text{by L'Hôpital's Rule}$$

Series converges when $\left| \dfrac{x-4}{6} \right| < 1$

$$|x-4| < 6$$

$$-6 < x-4 < 6$$

$$-2 < x < 10$$

When $x = -2$, series $= \displaystyle\sum_{n=1}^{\infty} \frac{(-6)^n}{n \cdot 6^n}$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n\, 6^n}{n \cdot 6^n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Converges by the Alternating Series Test

$\rightarrow$

When $x = 10$, series $= \sum_{n=1}^{\infty} \frac{6^n}{n \cdot 6^n}$

$$= \sum_{n=1}^{\infty} \frac{1}{n}$$

Diverges (p-series with $p=1$).

The interval of convergence is $-2 \leq x < 10$.

(24) $(1+x)^k \approx 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3$

$$(1+x^2)^k \approx 1 + kx^2 + \frac{k(k-1)}{2!}x^4 + \frac{k(k-1)(k-2)}{3!}x^6$$

$$(1+x^2)^{1/3} \approx 1 + \frac{x^2}{3} + \frac{1}{2}\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)x^4$$

$$+ \frac{1}{6}\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)x^6$$

$$\approx 1 + \frac{x^2}{3} - \frac{x^4}{9} + \frac{5x^6}{81}$$

(25) $\cos x \simeq 1 - \frac{x^2}{2} + \frac{x^4}{24}$

$\cos x^2 \simeq 1 - \frac{x^4}{2} + \frac{x^8}{24}$

$$\int_0^1 \cos x^2\,dx \simeq \int_0^1 \left[1 - \frac{x^4}{2} + \frac{x^8}{24}\right] dx$$

$$\simeq \left[x - \frac{x^5}{10} + \frac{x^9}{216}\right]_0^1$$

$$\simeq 0.9046$$