1. [3 marks] Let $z = y^2 e^{2x} - x^3 \sin 4y$. Find:

a) $\frac{\partial z}{\partial y}$ $= 2ye^{2x} - x^3(4\cos 4y)$

$= 2ye^{2x} - 4x^3 \cos 4y$

b) $\frac{\partial^2 z}{\partial y^2}$ $= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$

$= 2e^{2x} - 4x^3(-4\sin 4y)$

$= 2e^{2x} + 16x^3 \sin 4y$

c) $\frac{\partial^2 z}{\partial x \partial y}$ $= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$

$= 2y(2e^{2x}) - 12x^2 \cos 4y$

$= 4ye^{2x} - 12x^2 \cos 4y$

2. [6 marks] a) Evaluate $\int_0^9 \int_{\sqrt{x}}^3 (xy+1)dy\, dx$

$$= \int_0^9 \left[\frac{xy^2}{2} + y \right]_{y=\sqrt{x}}^{y=3} dx$$

$$= \int_0^9 \left[\frac{9x}{2} + 3 - \frac{x(\sqrt{x})^2}{2} - \sqrt{x} \right] dx$$

$$= \int_0^9 \left(\frac{9x}{2} + 3 - \frac{x^2}{2} - \sqrt{x} \right) dx$$

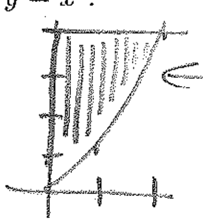
$$= \left[\frac{9x^2}{4} + 3x - \frac{x^3}{6} - \frac{2}{3}x^{3/2} \right]_0^9$$

$$= \frac{9(9)^2}{4} + 27 - \frac{9^3}{6} - \frac{2}{3}(9)^{3/2} - 0$$

$$= 69.75$$

b) Express the following volume as a double integral.
DO NOT EVALUATE the double integral.

The volume under $z = x^3 + y^3$ over the region bounded by $x = 0, y = 4$ and $y = x^2$.



$\leftarrow y = x^2$

$$x^2 \le y \le 4$$

$$0 \le x \le 2$$

$$V = \iint z\, dy\, dx$$

$$V = \int_0^2 \int_{x^2}^4 (x^3 + y^3)\, dy\, dx$$

3. [5 marks] Find an implicit solution if $y = 0$ when $x = \frac{\pi}{2}$:

$5y^3(\csc 2x)dy = 3(y^4 + 3)dx$

$$\frac{5y^3}{y^4+3}\,dy = \frac{3}{\csc 2x}\,dx$$ Separable

$$\frac{5}{4}\cdot\frac{4y^3}{y^4+3}\,dy = 3\sin 2x\,dx$$

$$\frac{5}{4}\int\frac{4y^3}{y^4+3}\,dy = \int 3\sin 2x\,dx$$

$$\boxed{\frac{5}{4}\ln(y^4+3) = \frac{-3\cos 2x}{2} + C}$$

Sub $y=0$, $x=\frac{\pi}{2}$:

$$\frac{5}{4}\ln 3 = -\frac{3\cos\pi}{2} + C$$

$$\frac{5}{4}\ln 3 = \frac{3}{2} + C$$

$$C = \frac{5}{4}\ln 3 - \frac{3}{2}$$

$$\frac{5}{4}\ln(y^4+3) = -\frac{3}{2}\cos 2x + \frac{5}{4}\ln 3 - \frac{3}{2}$$

4. [6 marks] Find an explicit solution (solve for y):

$dy - x^7 e^{4x} dx - \frac{6}{x} y dx = 0$

$$\boxed{dy - \frac{6}{x} y\,dx = x^7 e^{4x} dx}$$ Linear

$P(x) = -\frac{6}{x}$

$\int P(x)dx = -6 \ln x$

$I.F. = e^{-6\ln x} = e^{\ln x^{-6}} = x^{-6}$

$x^{-6} dy - 6x^{-7} y dx = x e^{4x} dx$

$\int (x^{-6} dy - 6x^{-7} y) dx = \int x e^{4x} dx$

	D	I
+	x	e^{4x}
−	1	$e^{4x}/4$
		$e^{4x}/16$

$x^{-6} y = \frac{xe^{4x}}{4} - \frac{e^{4x}}{16} + c$

Explicit:

$y = \frac{x^7 e^{4x}}{4} - \frac{x^6 e^{4x}}{16} + Cx^6$

5. [5 marks] Consider the intensity of light below the ocean's surface.

The rate of change of intensity I with respect to depth x is proportional to intensity. Let the intensity at the surface be A (a constant). The intensity at a depth of 4 meters is $0.7A$. What is the intensity at a depth of 18 meters?

Start with an appropriate DE and show all your work.

I: intensity

x: depth (m)

$$\frac{dI}{dx} \propto I$$

$$\frac{dI}{dx} = kI \quad (k \text{ constant})$$

$$\frac{dI}{I} = k\,dx$$

$$\int \frac{dI}{I} = \int k\,dx$$

$$\ln I = kx + C_1$$

$$e^{\ln I} = e^{kx + C_1}$$

$$I = e^{kx} \cdot e^{C_1}$$

$$\boxed{I = Ce^{kx}}$$

$\boxed{\text{Sub } x=0,\ I=A}$:

$$A = Ce^{0}$$

$$C = A$$

$$\boxed{I = Ae^{kx}}$$

$\boxed{\text{Sub } x=4,\ I=0.7A}$:

$$0.7A = Ae^{4k}$$

$$0.7 = e^{4k}$$

$$\ln 0.7 = 4k$$

$$k = \frac{\ln 0.7}{4}$$

$$\boxed{I = Ae^{\frac{\ln 0.7}{4}x}}$$

$\boxed{\text{Sub } x=18:}$

$$I = Ae^{\left[\frac{\ln 0.7}{4}(18)\right]}$$

$$I \simeq 0.2A$$

(20% of the intensity at the surface)