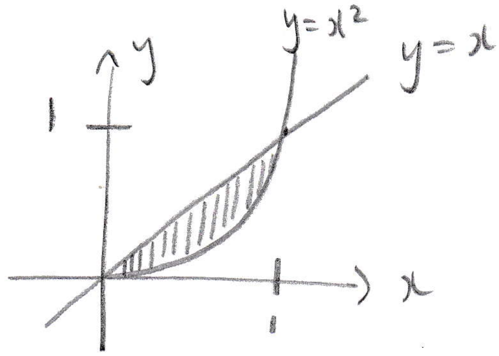
(14)



$$\boxed{\begin{gathered} x^2 \le y \le x \\ 0 \le x \le 1 \end{gathered}}$$

$$V = \iint z \, dy \, dx$$

$$= \int_0^1 \int_{x^2}^{x} z \, dy \, dx$$

$$= \int_0^1 \int_{x^2}^{x} xy \, dy \, dx$$

$$= \int_0^1 \left[x \cdot \frac{y^2}{2} \right]_{y=x^2}^{y=x} dx$$

$$= \int_0^1 \left[\frac{x^3}{2} - \frac{x(x^2)^2}{2} \right] dx$$

$$= \int_0^1 \left(\frac{x^3}{2} - \frac{x^5}{2} \right) dx$$

$$= \left[\frac{x^4}{8} - \frac{x^6}{12} \right]_0^1$$

$$= \left(\frac{1}{8} - \frac{1}{12} \right) - 0$$

$$= \frac{4}{96}$$

$$\text{or } \frac{1}{24}$$

(15) $\boxed{y = \sqrt{Cx - x^2}}$

$$y' = \frac{1}{2}\left(Cx - x^2\right)^{-1/2}(C - 2x)$$

$$\boxed{y' = \frac{C - 2x}{2\sqrt{Cx - x^2}}}$$

$$LS = 2xyy' + x^2$$

$$= 2x\sqrt{Cx - x^2} \cdot \frac{(C - 2x)}{2\sqrt{Cx - x^2}} + x^2$$

$$= x(C - 2x) + x^2$$

$$= Cx - 2x^2 + x^2$$

$$= Cx - x^2$$

$$= \sqrt{Cx - x^2}^{\,2}$$

$$= y^2$$

$$= RS$$

(16) $(\sin x)\,dy = x^3(\sin^2 x)\,dx + \frac{3y\sin x}{x}\,dx$

$$dy = x^3 \sin x\,dx + \frac{3y}{x}\,dx$$

$$\boxed{dy - \frac{3}{x}y\,dx = x^3 \sin x\,dx}$$

Linear

$P(x) = -\frac{3}{x}$

$\int P(x)\,dx = -3\ln x$

$I.F. = e^{-3\ln x} = e^{\ln x^{-3}} = x^{-3}$

$x^{-3}dy - 3x^{-4}y\,dx = \sin x\,dx$

Integrate:

$$\boxed{x^{-3}y = -\cos x + C}$$

Sub $y=1$, $x=\pi$: $\quad \frac{1}{\pi^3} = -(-1) + C$

$$C = \frac{1}{\pi^3} - 1$$

$$\boxed{x^{-3}y = -\cos x + \frac{1}{\pi^3} - 1}$$

(17) $6xy\,dx + (x^2+7)\,dy = 0$

Divide by $y(x^2+7)$:

$$\frac{6x\,dx}{x^2+7} + \frac{dy}{y} = 0$$ $\boxed{\text{Separable}}$

$$\int \frac{6x\,dx}{x^2+7} + \int \frac{dy}{y} = \int 0$$

$$3\int \frac{2x\,dx}{x^2+7} + \int \frac{dy}{y} = \int 0$$

$$3\ln(x^2+7) + \ln y = c_1$$ $\leftarrow$ Implicit Solution

$$\ln(x^2+7)^3 + \ln y = c_1$$

$$\ln\left[(x^2+7)^3 \cdot y\right] = c_1$$

$e^{LS} = e^{RS}$: $(x^2+7)^3 \cdot y = \cancel{e^{c_1}}\ c$

$$y = \frac{c}{(x^2+7)^3}$$ $\leftarrow$ Explicit Solution

(18) let T = coffee's temp ($°C$)
t = time (mins)

$T(0) = 90$
$T(5) = 80$
$T(15) = ?$

$$\frac{dT}{dt} = k(T-20)$$

$$\frac{dT}{T-20} = k\,dt$$

$$\int \frac{dT}{T-20} = \int k\,dt$$

$$\ln(T-20) = kt + C_1$$

$$T-20 = e^{kt+C_1} \leftarrow e^{kt} \cdot \cancel{e^{C_1}}\, C$$

$$T-20 = Ce^{kt}$$

$$\boxed{T = 20 + Ce^{kt}}$$

Sub $T=90$, $t=0$: $90 = 20 + C$

$C = 70$

$$\boxed{T = 20 + 70\,e^{kt}}$$

$\rightarrow$

Sub $T=80$, $t=5$:

$$80 = 20 + 70e^{5k}$$

$$60 = 70e^{5k}$$

$$\frac{60}{70} = e^{5k}$$

$$\ln\left(\frac{60}{70}\right) = 5k$$

$$k = \frac{1}{5}\ln\left(\frac{60}{70}\right)$$

$$\boxed{T = 20 + 70e^{\frac{1}{5}\ln\left(\frac{60}{70}\right)t}}$$

Sub $t=15$:

$$T = 20 + 70e^{\frac{1}{5}\ln\left(\frac{60}{70}\right)\cdot 15}$$

$$\approx 64\ ^\circ C$$

19 a) $y'' + 8y' + 7y = 0$

$m^2 + 8m + 7 = 0$

$(m+1)(m+7) = 0$

$m = -1, -7$ Distinct Real Roots

$y = C_1 e^{-x} + C_2 e^{-7x}$

b) $y'' + 8y' + 16y = 0$

$m^2 + 8m + 16 = 0$

$(m+4)^2 = 0$

$m = -4, -4$ Repeated Real Roots

$y = C_1 e^{-4x} + C_2 x e^{-4x}$

or $y = e^{-4x}(C_1 + C_2 x)$

c) $y'' + 8y' + 18y = 0$

$m^2 + 8m + 18 = 0$

$m = \dfrac{-8 \pm \sqrt{64 - 4.18}}{2}$

$m = \dfrac{-8 \pm \sqrt{-8}}{2} \leftarrow \sqrt{4}\sqrt{2}\sqrt{-1}$

$\rightarrow$

$$m = \frac{-8 \pm 2\sqrt{2}\,j}{2}$$

$$m = -4 \pm \sqrt{2}\,j$$

Complex Roots

$\alpha = -4 \quad \beta = \sqrt{2}$

$$y = e^{-4x}\left(C_1 \sin\sqrt{2}x + C_2 \cos\sqrt{2}x\right)$$

20 a) $y_p = Ax^2 + Bx + C$ not bad case

b) $\cancel{y_p = A\cos x + B\sin x}$ bad case

$y_p = Ax\cos x + Bx\sin x$

c) $y_p = Ae^{7x}$ not bad case

d) $y_p = Ax^2e^{4x} + Bxe^{4x} + Ce^{4x}$ not bad case

or $y_p = (Ax^2 + Bx + C)e^{4x}$

(21) $y'' + 7y' + 12y = \sin 2x, \quad y(0) = 4, \; y'(0) = -3$

1) Find y_c

$$m^2 + 7m + 12 = 0$$
$$(m+3)(m+4) = 0$$
$$m = -3, -4$$
$$y_c = C_1 e^{-3x} + C_2 e^{-4x}$$

2) Find y_p

$y_p = A\sin 2x + B\cos 2x$ not bad case

3) $y_p \to DE$

$$\begin{cases} y_p = A\sin 2x + B\cos 2x \\ y_p' = 2A\cos 2x - 2B\sin 2x \\ y_p'' = -4A\sin 2x - 4B\cos 2x \end{cases}$$

$$y'' + 7y' + 12y = \sin 2x$$
$$(-4A\sin 2x - 4B\cos 2x) + 7(2A\cos 2x - 2B\sin 2x) + 12(A\sin 2x + B\cos 2x) = \sin 2x$$

$$(-4A - 14B + 12A)\sin 2x + (-4B + 14A + 12B)\cos 2x = \sin 2x + 0\cos 2x$$

$\rightarrow$

$$\begin{cases} 8A - 14B = 1 & (1) \\ 14A + 8B = 0 & (2) \end{cases}$$

$8 \times (1)$: $64A - 112B = 8$

$14 \times (2)$ $\quad 196A + 112B = 0$

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$260A = 8$

$$A = \frac{8}{260} = \frac{2}{65}$$

$A = \frac{2}{65} \rightarrow (2)$: $\frac{28}{65} + 8B = 0$

$$8B = \frac{-28}{65}$$

$$B = \frac{-28}{520} = \frac{-7}{130}$$

$$y_p = \frac{2}{65}\sin 2x - \frac{7}{130}\cos 2x$$

4) $y = y_c + y_p$

$$\boxed{y = C_1 e^{-3x} + C_2 e^{-4x} + \frac{2}{65}\sin 2x - \frac{7}{130}\cos 2x}$$

5) Initial Conditions

$x = 0$, $y = 4$: $4 = C_1 + C_2 - \frac{7}{130}$

$$C_1 + C_2 = \frac{527}{130} \quad (3)$$

$\rightarrow$

$$y' = -3C_1e^{-3x} - 4C_2e^{-4x} + \frac{4}{65}\cos 2x + \frac{14}{130}\sin 2x$$

$$y'_{x=0} = -3 : \quad -3 = -3C_1 - 4C_2 + \frac{4}{65}$$

$$-3C_1 - 4C_2 = \frac{-199}{65} \quad (4)$$

$$3 \times (3) : \quad 3C_1 + 3C_2 = \frac{1581}{130}$$

$$(4) : \quad + \quad -3C_1 - 4C_2 = \frac{-199}{65}$$

$$-C_2 = \frac{1581}{130} - \frac{199}{65}$$

$$-C_2 = \frac{1183}{130}$$

$$\boxed{C_2 = -\frac{1183}{130}}$$

$$C_2 = \frac{-1183}{130} \rightarrow (3) : \quad C_1 - \frac{1183}{130} = \frac{527}{130}$$

$$\boxed{C_1 = \frac{1710}{130}}$$

$$y = \frac{1710}{130}e^{-3x} - \frac{1183}{130}e^{-4x} + \frac{2}{65}\sin 2x - \frac{7}{130}\cos 2x$$

(22) $m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx + f(t)$

$\boxed{\begin{aligned} x(0) &= 0.2 \\ x'(0) &= -0.3 \end{aligned}}$ ⊖ ⊕

$2\frac{d^2x}{dt^2} = -10\frac{dx}{dt} - 12x + \sin 3t$

$$\boxed{2\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 12x = \sin 3t}$$

1) Find x_c

$2n^2 + 10n + 12 = 0$

$n^2 + 5n + 6 = 0$

$(n+2)(n+3) = 0$

$n = -2, -3$

$x_c = C_1 e^{-2t} + C_2 e^{-3t}$ ← Note: x is a function of t

2) Find x_p

$x_p = A\sin 3t + B\cos 3t$ not bad case

3) $x_p \to$ DE

$$\begin{cases} x_p = A\sin 3t + B\cos 3t \\ x_p' = 3A\cos 3t - 3B\sin 3t \\ x_p'' = -9A\sin 3t - 9B\cos 3t \end{cases}$$

$$2x'' + 10x' + 12x = \sin 3t$$

$$2(-9A\sin 3t - 9B\cos 3t) + 10(3A\cos 3t - 3B\sin 3t)$$
$$+12(A\sin 3t + B\cos 3t) = \sin 3t$$

$$(-18A - 30B + 12A)\sin 3t + (-18B + 30A + 12B)$$
$$= \sin 3t + 0\cos 3t$$

$$\begin{cases} -6A - 30B = 1 & ① \\ 30A - 6B = 0 & ② \end{cases}$$

$$5 \times ①: \quad -30A - 150B = 5$$
$$②: + \quad 30A - 6B = 0$$

$$-156B = 5$$
$$B = \frac{-5}{156}$$

$$B = \frac{-5}{156} \to ②: \quad 30A + \frac{30}{156} = 0$$
$$30A = \frac{-30}{156}$$
$$A = \frac{-1}{156}$$

$$x_p = \frac{-1}{156}\sin 3t - \frac{5}{156}\cos 3t$$

$\rightarrow$

4) $x = x_c + x_p$

$$x = C_1 e^{-2t} + C_2 e^{-3t} - \frac{1}{156}\sin 3t - \frac{5}{156}\cos 3t$$

5) Initial Conditions

$x = 0.2$, $t = 0$: $\quad 0.2 = C_1 + C_2 - \frac{5}{156}$

$$C_1 + C_2 = \frac{181}{780} \quad (3)$$

$$x' = -2C_1 e^{-2t} - 3C_2 e^{-3t} - \frac{3}{156}\cos 3t + \frac{15}{156}\sin 3t$$

$x' = -0.3$, $t = 0$: $\quad -0.3 = -2C_1 - 3C_2 - \frac{3}{156}$

$$-2C_1 - 3C_2 = \frac{-219}{780} \quad (4)$$

$2 \times (3)$: $\quad 2C_1 + 2C_2 = \frac{362}{780}$

(4) : $\quad -2C_1 - 3C_2 = \frac{-219}{780}$

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$$-C_2 = \frac{143}{780}$$

$$\boxed{C_2 = \frac{-143}{780}}$$

$C_2 = \frac{-143}{780} \rightarrow (3)$: $\quad C_1 - \frac{143}{780} = \frac{181}{780}$

$$\boxed{C_1 = \frac{324}{780}}$$

$$x = \frac{324}{780} e^{-2t} - \frac{143}{780} e^{-3t} - \frac{1}{156}\sin 3t - \frac{5}{156}\cos 3t$$