# 26.6 Applications

$W = F \cdot d$ when force is constant

More generally:

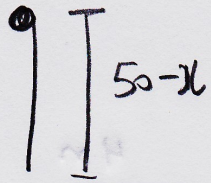
$$W = \int_a^b F(x)\,dx$$

$F(x)$: force applied at distance $x$

Ex: Find the work done in winding up 30m of a 50m chain that weighs 2 N/m.

Let $x$ = length of chain already wound up.

$0 \le x \le 30$




$x$ must be increasing to avoid a sign error

$F(x) = 2(50 - x)$ N/m · m

$= 100 - 2x$ N

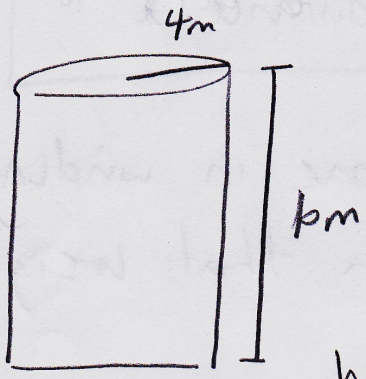
$$W = \int_0^{30} F(x)\,dx \qquad \text{(N} \leftarrow \text{m)}$$

$$= \int_0^{30} (100 - 2x)\,dx$$

$$= \left[100x - x^2\right]_0^{30}$$

$$= 2100 \text{ N·m} \quad \text{or} \quad 2100 \text{ J}$$

Ex:



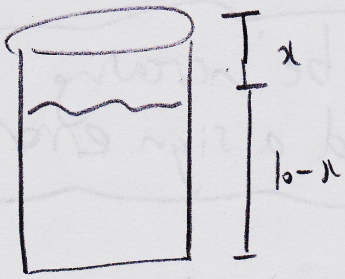

Cylindrical tank, initially full of water.

Water weighs 9800 N/m³.

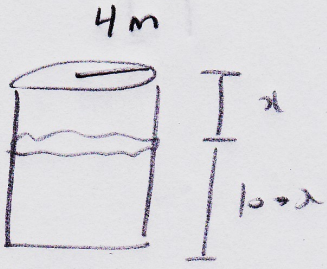
Work done in pumping out the tank?

Let $x$ = height of water already pumped out

$0 \le x \le 10$




Imagine a disk of water: $dx$ thick




$V_{disk}$ = π · radius² · thickness

$= \pi 4^2\,dx$

Work = ~~Force~~ Weight · distance

$= 9800\ N/m^3 \cdot$ Volume · distance

$dW = 9800\ (\pi \cdot 4^2\, dx)\, x$

$dW = 9800(16)\pi\, x\, dx$

$W = \int_0^{10} 9800(16)\,\pi\, x\, dx$

$= 9800(16)\,\pi\, \frac{x^2}{2}\Big|_0^{10}$

$= 2.5 \times 10^7\ J$

Ex: Gravitational attraction between 2 objects is $F = \frac{k}{x^2}$, where $x$ = distance between objects. Find work done in moving the objects from 20m apart to 40m apart.

$x$ = distance between objects

$20 \le x \le 40$

$W = \int_{20}^{40} F(x)\, dx$

$$= \int_{20}^{40} \frac{k}{x^2}\, dx$$

$$= -k x^{-1} \Big|_{20}^{40}$$

$$= -0.025k + 0.05k$$

$$= 0.025k \text{ J}$$

Ex: A spring of natural length 1m requires a force of 8N to stretch it to 3m. Find the work done in stretching it to 3.5m.

2.5m of stretch

2m of stretch

$$F = kx$$

$$8 = k(2)$$

$$k = 4 \text{ N/m} \qquad \boxed{F = 4x}$$

$x$ = amount of stretch

$0 \le x \le 2.5$

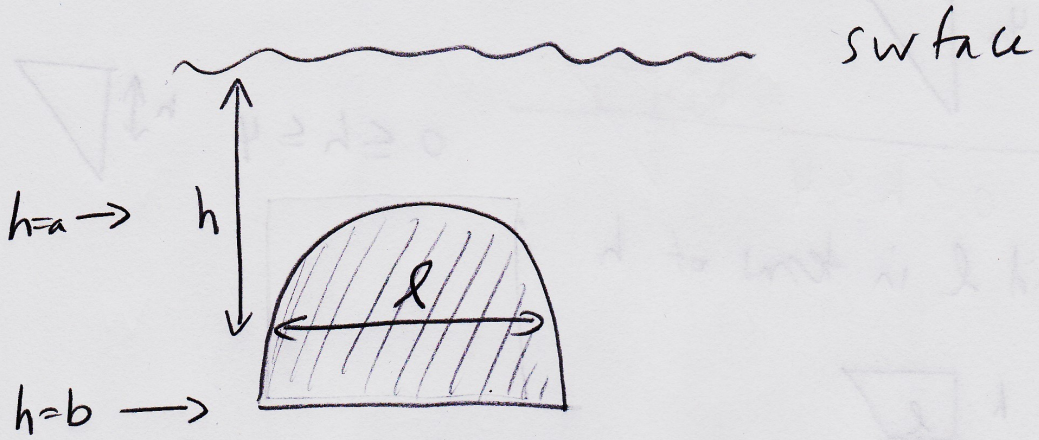
$$W = \int F(x)\, dx$$

$$= \int_0^{2.5} 4x\, dx$$

$$= \left[2x^2\right]_0^{2.5}$$

$$= 12.5 \text{ J}$$

# Force on Vertical Plate due to Water Pressure

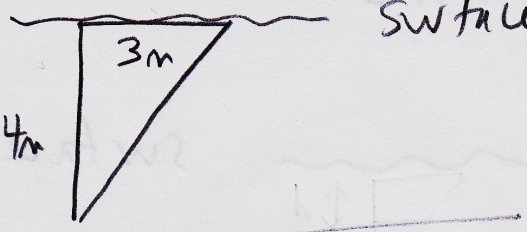



$$F = 9800 \int_a^b \ell h \, dh \quad (N)$$

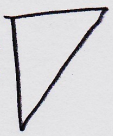
$\ell$ : $m^3$

$h$ : $N/m^3$

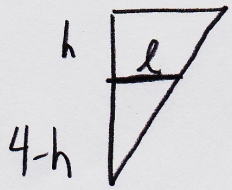
Ex: Find total force on plate:




$0 \le h \le 4$




Find $\ell$ in terms of $h$:




Similar triangles: $\frac{3}{4} = \frac{\ell}{4-h}$

$$\ell = \frac{3}{4}(4-h)$$

$$F = 9800 \int_a^b \ell h \, dh$$

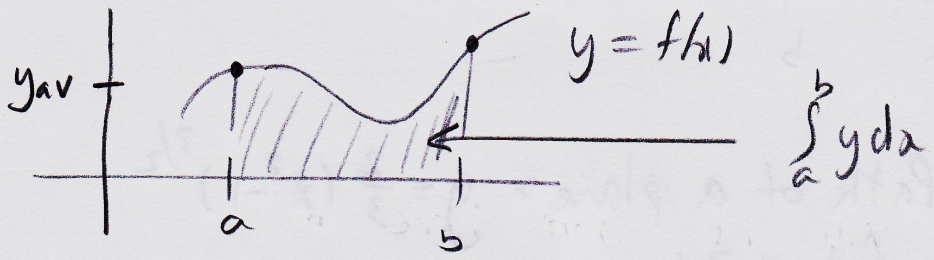
$$= 9800 \int_0^4 \frac{3}{4}(4-h)h \, dh$$

$$= 7350 \int_0^4 (4h - h^2) \, dh$$

$$= 7350 \left[ 2h^2 - \frac{h^3}{3} \right]_0^4 = 7350\left(\frac{32}{3}\right) = 78400 \text{ N}$$

# Average value of a function

$$y_{av} = \frac{\int_a^b y\,dx}{b-a}$$




**Ex:** Find average value of $y = x^4$ from $x = 0$ to $x = 2$

$$y_{av} = \frac{\int_0^2 x^4\,dx}{2-0}$$

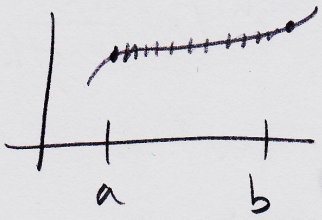
$$= \frac{1}{2}\int_0^2 x^4\,dx$$

$$= \frac{1}{2}\left[\frac{x^5}{5}\right]_0^2$$

$$= \frac{1}{2}\left(\frac{32}{5}\right)$$

$$= \frac{16}{5}$$

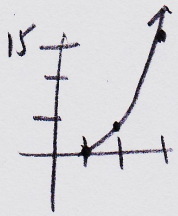
# Arc Length




$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Ex: Path of a plane: $y = \frac{2}{3}(x^2-1)^{3/2}$ for $x \geq 1$.

Find length of path for $1 \leq x \leq 3$




$$\frac{dy}{dx} = \frac{2}{3} \cdot \frac{3}{2}(x^2-1)^{1/2}(2x)$$

$$= \sqrt{x^2-1}\,(2x)$$

$$\left(\frac{dy}{dx}\right)^2 = (x^2-1)(4x^2)$$

$$= 4x^4 - 4x^2$$

$$s = \int_1^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_1^3 \sqrt{4x^4 - 4x^2 + 1}\, dx$$

$$= \int_1^3 \sqrt{(2x^2-1)^2}\,dx$$

$\sqrt{\#^2} = \#$ when $\# > 0$

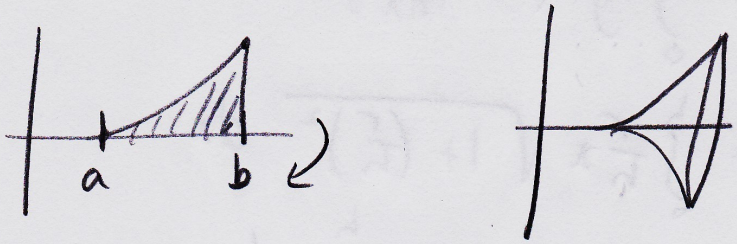
$$= \int_1^3 (2x^2-1)\,dx$$

$$= \left[\frac{2x^3}{3} - x\right]_1^3$$

$$= \left(\frac{54}{3} - 3\right) - \left(\frac{2}{3} - 1\right)$$

$$= \frac{52}{3} - 2$$

$$= \frac{46}{3}$$

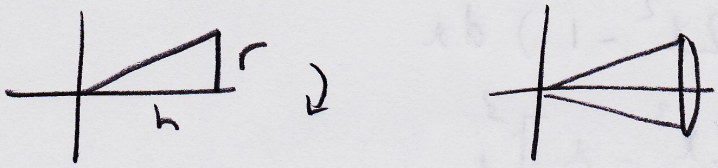





Surface area of Solid of Revolution

$$S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

**Ex:** Find surface area of a cone with height $h$ and radius $r$.




Line $y = mx + b$

$y = \frac{r}{h}x + 0$

$$\boxed{y = \frac{r}{h}x}$$

$\frac{dy}{dx} = \frac{r}{h}$

$$S = 2\pi \int_0^h y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_0^h \frac{r}{h}x \sqrt{1 + \left(\frac{r}{h}\right)^2}\, dx$$

$$= 2\pi \frac{r}{h} \sqrt{1 + \left(\frac{r}{h}\right)^2} \int_0^h x\, dx$$

$$= 2\pi \frac{r}{h} \sqrt{1 + \left(\frac{r}{h}\right)^2} \left[\frac{x^2}{2}\right]_0^h$$

$$= 2\pi \frac{r}{h} \sqrt{1 + \frac{r^2}{h^2}} \left(\frac{h^2}{2}\right)$$

$$= \pi r h \sqrt{1 + \frac{r^2}{h^2}}$$

$\sqrt{h^2}$

$$= \pi r \sqrt{h^2 + r^2}$$