24.8 Differentials and Linear Approximation

Ex: $y = 3x^6 + 7x^3$

$$\frac{dy}{dx} = 18x^5 + 21x^2$$

$$dy = (18x^5 + 21x^2)\,dx$$

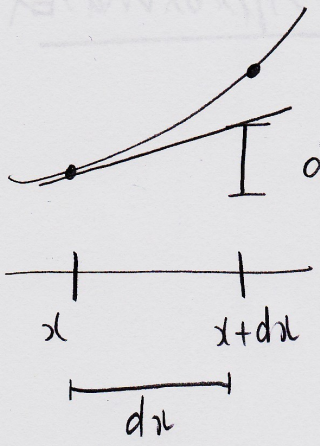
dy: differential of y — dx: differential of x

Ex: Find the differential of V

$$V = \frac{4}{3}\pi r^3$$

$$dV = 4\pi r^2\,dr$$

Application: Error calculation



Δx or dx: small change in x

dy: rise of tangent line

Δy: change in y

When dx is small, $\Delta y \approx dy$

<u>Caution</u>: Say $y = x^2$

x is the independent variable

Δx and dx mean the same thing

y is the dependent variable

Δy and dy are different

Ex: $f(x) = x^2 + 2$

Compute Δy and dy for $x = 1$ and $dx = 0.05$

$$dy = f'(x)\,dx$$

$$dy = 2x\,dx$$

$$dy = 2(1)(0.05) = 0.1$$

↑ approx. change in y

$$\Delta y = f(x + dx) - f(x)$$

$$= f(1.05) - f(1)$$

$$= (1.05^2 + 2) - 3$$

$$= 0.1025$$

↑ true change in y

Ex: Approximate Δy using dy

$y = x^{1/2}$ $\quad x = 4$ $\quad dx = 0.1$

$$\Delta y \approx dy$$

$$= \frac{1}{2} x^{-1/2}\,dx$$

$$= \frac{1}{2}(4)^{-1/2}(0.1)$$

$$= 0.025$$

Ex: Sphere's radius is measured to be 2.500cm. If the true radius is 2.512 cm, estimate the error in volume.

$r = 2.5$ (measured)

dr = error in radius
= true - measured
= 0.012

$$V = \frac{4}{3}\pi r^3$$

$$dV = 4\pi r^2 dr$$

error in volume $= 4\pi(2.5)^2(0.012)$

$\approx 0.9425\ cm^3$

Why use differentials?

Absolute error in radius: dr

Relative error in radius: $\frac{dr}{r}$

Relative error in volume: $\frac{dV}{V} = \frac{4\pi r^2 dr}{\left(\frac{4}{3}\pi r^3\right)}$

$= 3\frac{dr}{r}$

Comes from dimension of sphere

No predictable relationship between $\frac{\Delta V}{V}$ and $\frac{\Delta r}{r}$

Ex: Given $A = \pi r^2$ $\frac{dr}{r} = 0.02$

Find $\frac{dA}{A}$

$$\frac{dA}{A} = \frac{2\pi r\, dr}{\pi r^2}$$

$$= \frac{2\,dr}{r}$$

$$= 2(0.02)$$

$$= 0.04$$

2% relative error in radius of circle

$\Rightarrow$ 4% " area

Ex: Given $F = \frac{k}{r^2}$, k constant

Show $\frac{dF}{F} = -2\frac{dr}{r}$

$\rightarrow F = kr^{-2}$

$$\frac{dF}{F} = \frac{-2kr^{-3}dr}{kr^{-2}}$$

$$= \frac{-2\,dr}{r}$$

$$= -2\frac{dr}{r}$$

Let a = # near x such that $f(a)$ is known

$$f(x) = f(a) + \text{error}$$

$$\boxed{f(x) \approx f(a) + f'(a)(x-a)}$$

"linearization of $f(x)$ at $x=a$"

$x-a$: small change in x

Valid near $x=a$

Ex: a) Find the linearization of $f(x) = x^4$ at $x = 2$

$\boxed{a=2}$

$f(x) \approx f(2) + f'(2)(x-2)$ $\qquad f'(x) = 4x^3$

$\boxed{f(x) \approx 16 + 32(x-2)}$ (Valid near $x=2$)

b) Approximate 1.98^4 using part a)

$\boxed{x = 1.98}$

$1.98^4 \approx 16 + 32(1.98-2)$ ← easy to calculate

≈ 15.36

Ex: Estimate $\sqrt[3]{8.3}$ using a linearization (a.k.a. lin approximation)

$f(x) = x^{1/3}$

a? a: near 8.3 and f(a) known

$a = 8$

$f(x) \approx f(a) + f'(a)(x-a)$

$f(x) \approx f(8) + f'(8)(x-8)$

$f'(x) = \frac{1}{3}x^{-2/3}$

$f'(8) = \frac{1}{3}(8)^{-2/3} = \frac{1}{12}$

$f(x) \approx 2 + \frac{1}{12}(x-8)$

(Valid near $x=8$)

$f(8.3) \approx 2 + \frac{1}{12}(8.3-8)$

$= 2.025$

Application of Linearization (beyond Math 191)

$f(x) = x^4$ ↓ $f'(x) = 4x^3$ ↑ area, volume etc.

Mean, rate of change

$f'(x) = 4x^3 \Rightarrow f(x) = x^4 + C$ easy ish

$f'(x) = e^{x^2} \Rightarrow f(x) = ?$ hard

Use linearization to simplify $f'(x)$