

MAXIMIZING OR MINIMIZING A FUNCTION

Example 1. Find two nonnegative numbers that sum to 50 and whose product is maximized.

Example 2. Cut the corners from a $15\text{cm} \times 15\text{cm}$ metal sheet to form an open-topped box. Find the height of the box that maximizes its volume. What is the maximum volume?

Example 3. An animal pen is built with five pieces of fencing: four pieces enclosing a rectangular area and a fifth piece dividing the pen in two. Given 400m of fencing, what is the maximum area that can be enclosed? What are the dimensions of the pen?

Example 4. Find the point on the line $y = 2x + 3$ that is closest to the point $(2, 1)$.

Example 5. A closed rectangular box has a square base. The material for the top and sides of the box costs $\$1/\text{cm}^2$. The material for the base costs $\$2/\text{cm}^2$. Find the dimensions that maximize the volume of the box if the total cost must be $\$144$.

Example 6. A cylinder has volume 100cm^3 . What dimensions minimize its surface area?

24.7 Maximizing or Minimizing a Function

① Let $x = 1^{\text{st}}$ number
 $y = 2^{\text{nd}}$ number

1) Maximize $f = xy$

Substitution: $x+y=50$

2) Use restriction to get f in terms
of one variable

$$y=50-x \rightarrow f$$

$$f(x)=x(50-x)$$

$$\text{or } f(x)=50x-x^2$$

3) Set $f'(x)=0$

$$f'(x)=50-2x$$

$$50-2x=0$$

$$x=25$$

Note: To check that $x=25$ is a maximum:
"2nd derivative test"

$$f''(x)=-2$$

$$f''(25) < 0$$

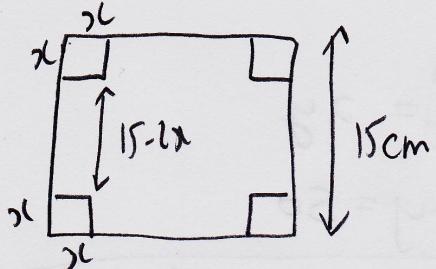
∴ $f(x)$ is CD at $x=25$
max ✓

4) Answer

$$x = 25$$

$$y = 50 - x = 25$$

(2)



1) Maximize $V = lwh$

$$V = (15-2x)^2 x$$

2) Single variable ~

3) Set $V' = 0$

$$V' = (15-2x)^2 + x \cdot 2(15-2x)(-2)$$

$$(15-2x)^2 - 4x(15-2x) = 0$$

$$(15-2x)(15-2x-4x) = 0$$

$$\overrightarrow{15-2x=0}$$

$$x = 7.5$$

$$\overrightarrow{15-6x=0}$$

$$x = \frac{15}{6} = 2.5$$

nonsense

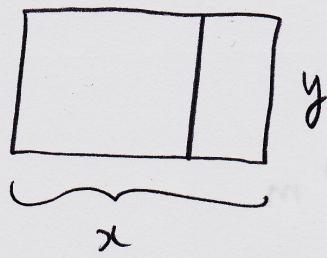
$$l=w=0$$

4) Answer

$$x = 2.5 \text{ cm}$$

$$\text{Max volume is } (15-2x)^2 x = 250 \text{ cm}^3$$

(3)



1) Maximize : $A = xy$

Restriction : $2x + 3y = 400$

2) Single variable

$$3y = 400 - 2x$$

$$y = \frac{400 - 2x}{3}$$

$$A = x \cdot \frac{(400 - 2x)}{3}$$

3) Set $A' = 0$

$$A' = \frac{400 - 2x}{3} + x \left(-\frac{2}{3}\right)$$

$$\frac{400 - 2x - 2x}{3} = 0$$

$$400 - 4x = 0$$

$$x = 100$$

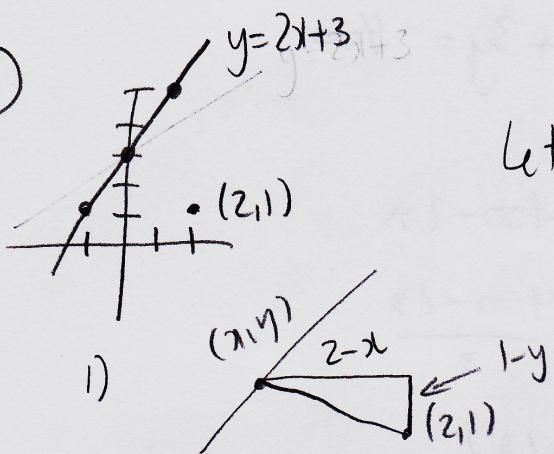
4) Answer

$$x = 100 \text{ m}$$

$$y = \frac{400 - 2x}{3} = \frac{200}{3} \text{ m}$$

$$\text{Area} = \frac{20,000}{3} \text{ m}^2$$

(4)



Let the point be (x, y)

$$\text{Minimize } d = \sqrt{(2-x)^2 + (1-y)^2}$$

Equivalent : minimize $d^2 = (2-x)^2 + (1-y)^2$
(gives same point (x, y))

Restriction : point on line $y = 2x + 3$

2) Single variable

$$f = d^2 \mid y = 2x + 3$$

$$f = (2-x)^2 + (1-(2x+3))^2$$

$$f = (2-x)^2 + (-2-2x)^2$$

3) Set $f' = 0$

$$2(2-x)(1) + 2(+2-2x)(-2) = 0$$

$$-4+2x + 8 + 8x = 0$$

$$4x + 4 = 0$$

$$x = -0.4$$

4) Answer

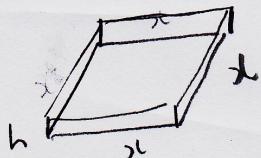
$$x = -0.4$$

$$y = 2x + 3 = 2.2 \quad \text{Point } (-0.4, 2.2)$$

Note: minimum distance is

$$d = \sqrt{(2-x)^2 + (1-y)^2} \approx 2.7 \text{ units}$$

(5)



1) Maximize $V = x^2h$

Restriction: $144 = 2x^2 + 4 \cdot x \cdot h + 2x^2 \quad \$ = \text{}/\text{cm}^2 \cdot \text{cm}^2$

top \nearrow 4 sides \nearrow bottom

2) Single variable

$$h = \frac{144 - 3x^2}{4x}$$

$$V = \frac{x^2 \cdot (144 - 3x^2)}{4x} \quad 0 = 7 + 8x \quad (8)$$

$$V = \frac{144(1x + 3x^3)}{4} + 8 + x^2 + 4$$

3) $V' = 0$

$$\frac{144 - 9x^2}{4} = 0$$

$$144 - 9x^2 = 0$$

$$x^2 = 16$$

$$x = \pm 4$$

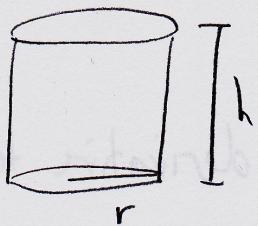
$$x = 4 \quad (\text{length} > 0)$$

4) Answer

$$x = 4 \text{ cm}$$

$$h = \frac{144 - 3x^2}{4x} = 6 \text{ cm}$$

(6)



$$V_{\text{cylinder}} = \pi r^2 h$$

$$SA_{\text{cylinder}} = 2\pi r^2 + 2\pi r h$$

top and
bottom side

1) Minimize $f = 2\pi r^2 + 2\pi r h$

Restriction: $\pi r^2 h = 100$

2) Single Variable

$$h = \frac{100}{\pi r^2}$$

$$f = 2\pi r^2 + 2\pi r \cdot \frac{100}{\pi r^2}$$

$$f = 2\pi r^2 + \frac{200}{r}$$

3) $f' = 0$

$$f' = 4\pi r - \frac{200}{r^2}$$

$$4\pi r - \frac{200}{r^2} = 0$$

$$4\pi r = \frac{200}{r^2}$$

$$r^3 = \frac{50}{\pi}$$

$$r = \sqrt[3]{\frac{50}{\pi}} \approx 2.52 \text{ cm}$$

Note: Could check using 2nd derivative test

$$f'' = 4\pi + \frac{400}{r^3}$$

$$f''(\sqrt[3]{\frac{50}{\pi}}) > 0 \Rightarrow f \text{ is concave up}$$



minimum ✓

4) Answer

$$r = \sqrt[3]{\frac{50}{\pi}} \approx 2.52 \text{ cm}$$

$$h = \frac{100}{\pi r^2} = \frac{100}{\pi \left(\frac{50}{\pi}\right)^{2/3}} \approx 5.03 \text{ cm}$$