RELATED RATES

1. A 40-foot ladder leans against a wall, with its base sliding away from the wall at 2 feet/min. How fast is the height of the ladder changing when the base is 24 feet from the wall?

2. When a spherical balloon is inflated, its radius increases by 3 cm/s. At what rate is the volume changing when the surface area is 100π cm^2?

3. How fast is the slope of the tangent line to $y = \frac{4}{2+5x}$ changing when $x = 2$ if x is increasing by 0.5 units/s?

4. Sand is filled into a large cone with radius 5m and height 10m to form a small cone of sand. If sand is filled at a rate of 10m^3/s, at what rate is the sand's radius increasing when it is 2m?

5. A man 1.8m tall walks with speed 2m/s away from a streetlight. If the streetlight sits atop a 6m pole, how fast is the tip of the man's shadow moving along the ground?

24.4 Related Rates

Variables x, y, r etc. all depend on time

<u>Ex</u>: $V = \frac{4}{3}\pi r^3$

r depends on t

Find $\frac{dV}{dt}$

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} \quad \text{Chain Rule}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

<u>Ex</u>: x and y depend on t

Find $\frac{d}{dt}[x^2 + y^2]$

$$= 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

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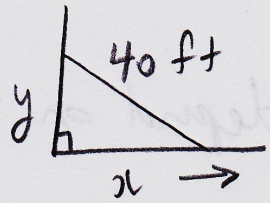


$\frac{dx}{dt} = 2$ ft/min

rate of change of x w.r.t. time

positive because x is increasing

$\frac{dy}{dt} = ?$ when $x = 24$ ft

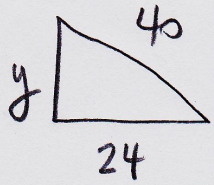
1) Equation

$$x^2 + y^2 = 40^2$$

2) Take $\frac{d}{dt}$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

3) Find any missing values



$$y^2 = 40^2 - 24^2$$

$$y = 32$$

4) Solve

$$2(24)(2) + 2(32)\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -1.5 \text{ ft/min}$$

negative because y is decreasing

(2) $\frac{dr}{dt} = 3\ cm/s$

$\frac{dV}{dt} = ?$

$SA = 100\pi\ cm^2$

$$\boxed{\begin{aligned} V_{sphere} &= \tfrac{4}{3}\pi r^3 \\ SA_{sphere} &= 4\pi r^2 \end{aligned}}$$

1) Equation

$V = \frac{4}{3}\pi r^3$

2) Take $\frac{d}{dt}$

$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

3) Missing values?

Given $SA = 100\pi$

$4\pi r^2 = 100\pi$

4) Solve

$\frac{dV}{dt} = 100\pi(3)$

$= 300\pi\ cm^3/s$

Know all formulas from class:
V, SA of sphere etc.

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(3) y' = slope of tangent line

$\frac{d}{dt}[y'] = ?$ $\quad x = 2 \quad \frac{dx}{dt} = 0.5$ units/s

1) Equation

$$y = 4(2+5x)^{-1}$$

$$y' = -4(2+5x)^{-2} \cdot 5$$

$$= -20(2+5x)^{-2}$$

2) Take $\frac{d}{dt}$

$$\frac{dy'}{dt} = \frac{dy'}{dx}\frac{dx}{dt}$$

$$= y''\frac{dx}{dt}$$

$$= 40(2+5x)^{-3} \cdot 5 \quad \frac{dx}{dt}$$

3) Solve

$$\frac{dy'}{dt} = 40(12)^{-3} \cdot 5\,(0.5)$$

$$\simeq 0.06 \quad \text{units/s}$$

Note: slope has no natural units

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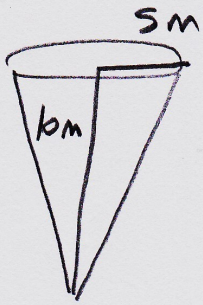
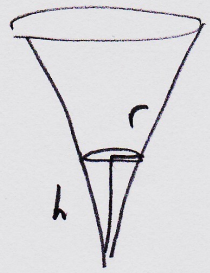




$$\boxed{V_{cone} = \frac{\pi}{3} r^2 h}$$

$\frac{dV}{dt} = 10\,m^3/s$ $\quad \frac{dr}{dt} = ?$ $\quad r = 2m$

1) Equation

$$V = \frac{\pi}{3} r^2 h$$

Similar triangles to eliminate h

$$\frac{10}{5} = \frac{h}{r}$$

$$h = 2r$$

$$V = \frac{2\pi}{3} r^3$$

2) $\frac{d}{dt}$

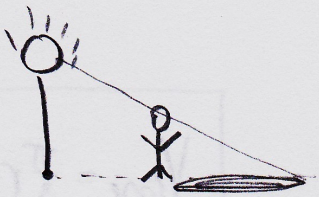
$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}$$

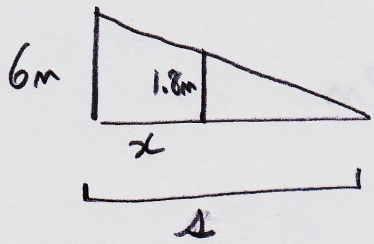
$$\frac{dV}{dt} = 2\pi r^2 \frac{dr}{dt}$$

3) Solve

$$10 = 8\pi \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = 0.4 \text{ m/s}$$

(5)





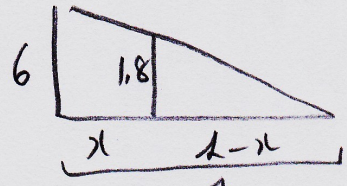
x = dist from pole to man

s = " shadow's tip

$\frac{dx}{dt} = 2$ m/s

Want $\frac{ds}{dt}$ ← rate of change of distance from shadow's tip to a stationary object (pole)

1) Equation



Similar triangles

$$\frac{6}{s} = \frac{1.8}{s-x}$$

$$6(s-x) = 1.8s$$

$$4.2s = 6x$$

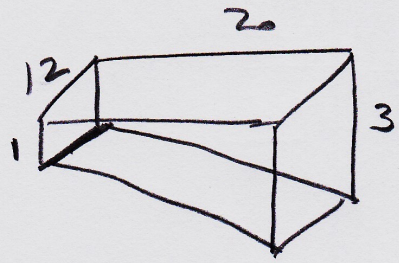
2) $\frac{d}{dt}$

$$4.2\frac{ds}{dt} = 6\frac{dx}{dt}$$

3) Solve $\quad 4.2\frac{ds}{dt} = 12 \Rightarrow \frac{ds}{dt} = 2.9$ m/s

Ex:

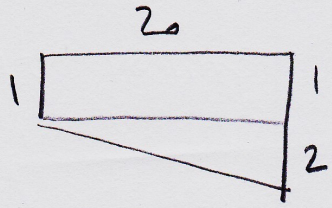
A 12m x 20m swimming pool is filled at a rate of $4\ m^3/min$. Ends are 1m and 3m deep, with a constant slope along the bottom. Rate of change of water depth when depth = 1.5m?

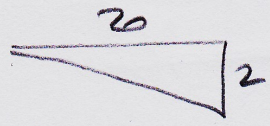


$$\frac{dV}{dt} = 4$$

Let y = depth

$$\frac{dy}{dt} = ? \qquad y = 1.5$$





Similar triangles

$$\frac{20}{2} = \frac{x}{y}$$

$$x = 10y$$

$V =$ triangular area $\cdot$ width of pool

$= \frac{1}{2} xy \, (12)$

$= 6xy$

$= 60y^2$ $\quad \boxed{x = 10y}$

$$\frac{dV}{dt} = 120y \frac{dy}{dt}$$

$$4 = 120(1.5) \frac{dy}{dt}$$

$$\frac{dy}{dt} \simeq 0.02 \text{ m/min}$$