# Math 191

## Practice Questions

1. Evaluate the following limits.

   (a) $\lim_{x\to 3} \dfrac{x^2 - x - 6}{x^2 - 4x + 3}$

   (b) $\lim_{x\to -2} \dfrac{x^2 - 4}{x^2 + 5x + 6}$

   (c) $\lim_{x\to 9} \dfrac{\sqrt{x} - 3}{x - 9}$

   *Answers:* (a) $5/2$ (b) $-4$ (c) $1/6$

2. (a) What is the definition of the derivative of a function?

   (b) Use the definition of the derivative to find $f'(x)$ for

   $$f(x) = 3x - 4x^2 + 1.$$

   *Answers:*

   (a)

   $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

   (b) After several algebraic steps, we should get $f'(x) = 3 - 8x$.

3. Find the slope of the tangent line to the curve $y = 3x^2 - \sqrt{x} - 10x$ at point $(4, 6)$.

   *Answer:* $55/4$

4. Find all the points at which a tangent line to the curve $y = 2x^3 + 1$ is perpendicular to the line of equation $x + 6y = 12$.

   *Answers:* $(1, 3)$ and $(-1, -1)$

5. Find all the points at which a tangent line to the curve $y = x^3 - 14x$ is parallel to the line of equation $4x + 2y = 1$.

   *Answers:* $(2, -20)$ and $(-2, 20)$

6. Find the derivative of the following functions. Simplify your answers.

   (a) $y = \dfrac{1}{\sqrt[3]{x}} + \dfrac{1}{3x+1} + 2x^5$

   (b) $y = \sqrt{4x^2 + 9}$

   (c) $y = 2x^4(x^3 + 1)^5$

   (d) $y = \dfrac{3x}{(2x+1)^2}$

   *Answers:*

   (a) $y' = \frac{-1}{3x^{4/3}} - \frac{3}{(3x+1)^2} + 10x^4$

   (b) $y' = \frac{4x}{\sqrt{4x^2+9}}$

(c) $y' = 2x^3(x^3+1)^4(19x^3+4)$

(d) $y' = \frac{3-6x}{(2x+1)^3}$

7. Find all the points on the curve

$$y = \frac{3x}{x^2+4}$$

where there is a horizontal tangent line?

*Answers:* $(2, 3/4)$ and $(-2, -3/4)$

8. The position $s$ of an object (in meters) after $t$ seconds is given by

$$s = \sqrt{2t^2+5}.$$

Find the instantaneous velocity of the object at $t = 3$ seconds.

*Answer:* 1.25 m/s

9. The resistance $R$ (in $\Omega$) of a certain wire as a function of the temperature $T$ (in °C) is given by

$$R = 12.0 + 0.35T + 0.0125T^2.$$

Find the instantaneous rate of change of $R$ with respect to $T$ when $T = 110$°C.

*Answer:* $\frac{dR}{dT} = 3.1\ \frac{\Omega}{°\text{C}}$

10. Find the second-derivative of the following function.

$$f(x) = \frac{1}{(x^2+1)^5}$$

*Answer:* $f''(x) = \frac{110x^2-10}{(x^2+1)^7}$

11. Find the second-derivative of

$$f(x) = \sqrt{x^3+1}$$

evaluated at $x = 2$.

*Answer:* 2/3

12. Use implicit differentiation to find $y' = \dfrac{dy}{dx}$ given that

$$x^3 + y^3 = xy + 1.$$

*Answer:* $y' = \frac{y-3x^2}{3y^2-x}$

13. Find the slope of the tangent line to the curve

$$(y^2+1)^4 = x^2y + 5x + 2.$$

at point $(2, 1)$.

*Answer:* 3/20

14. Find an equation for the normal line to the graph of $y = 3x - 4x^2$ at the point corresponding to $x = 1$.

    *Answer:* $y = \frac{x}{5} - \frac{6}{5}$

15. Find an equation for the tangent line at point $(2, 4)$ on the curve $x^3 + y^3 = 9xy$.

    *Answer:* $y = \frac{4}{5}x + \frac{12}{5}$

16. Find the magnitude and direction of the vector velocity when $t = 2$ of an object moving in the plane with $x$ and $y$ coordinates of position given by $x = \sqrt{1 + 4t}, \quad y = t^2 - 3t$. (The units of $t$ is seconds and $x, y$ are in meters.)

    *Answer:* $v = \frac{\sqrt{13}}{3} \approx 1.20$ m/s, $\quad \theta_v \approx 56.3°$ (in standard position)

17. Air is being pumped into a spherical balloon so that its volume increases at a rate of $100\,\text{cm}^3/\text{s}$. How fast is the radius of the balloon increasing when the diameter of the balloon is 20 cm?

    *Answer:* 0.08 cm/s

18. A ladder 5 meters long is leaning against a vertical wall. If the bottom of the ladder is pulled away at the constant rate of 2 m/s, how fast is the top of the ladder moving down the wall when the bottom is 3 meters from the wall?

    *Answer:* 1.5 m/s

19. A water tank has the shape of an inverted circular cone with a top radius of 2 m and a height of 6 m. If water is being pumped into the tank at a rate of $4\,\text{m}^3/\text{min}$, find the rate at which the water level is rising when the water is 4 m deep. *Hint:* $V = \frac{1}{3}\pi r^2 h$.

    *Answer:* 0.716 m/min

20. Suppose the resistance of a certain resistor varies with temperature according to

$$R = 0.2 + \frac{T^2}{50}$$

    where $T$ is in °C and $R$ in Ω. If the temperature is increasing at the rate of 0.1°C/s, find the rate of change of resistance when $T = 100$°C

    *Answer:* 0.4 Ω/s

21. Use Newton's method to estimate the first three decimals of the root of $f(x) = x^4 - 2x^2 - 1$ located inside the interval $1 \le x \le 2$.

    *Answer:* 1.553

22. Let $f(x) = x^4 - 2x^2 - 1$.
    (a) Find the intervals on which $f$ is increasing and decreasing.
    (b) Locate all relative maximum and relative minimum.
    (c) Find the intervals where $f$ is concave up and concave down and locate all inflection points.
    (d) Sketch the graph of $y = f(x)$.

    *Answers:*

(a)

| | $(-\infty,-1)$ | $(-1,0)$ | $(0,1)$ | $(1,+\infty)$ |
|---|---|---|---|---|
| $f(x)$ | $\searrow$ | $\nearrow$ | $\searrow$ | $\nearrow$ |

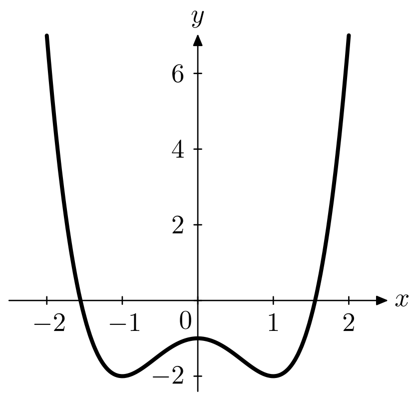
(b) The points $(-1,-2)$ and $(1,-2)$ are relative minimum and point $(0,-1)$ is a relative maximum.

(c)

| | $(-\infty,-\frac{1}{\sqrt{3}})$ | $(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})$ | $(\frac{1}{\sqrt{3}},+\infty)$ |
|---|---|---|---|
| $f(x)$ | $\cup$ | $\cap$ | $\cup$ |

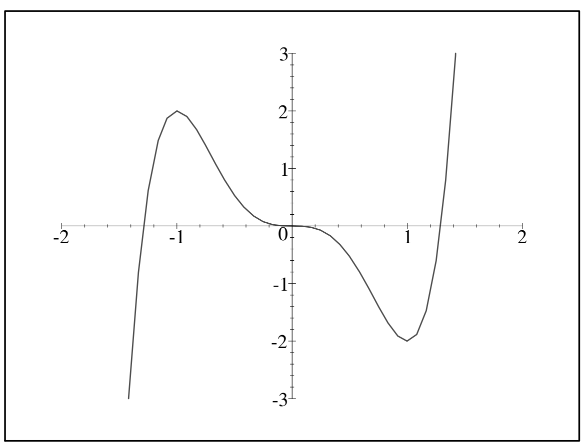
Both points $(\frac{1}{\sqrt{3}},-\frac{14}{9})$ and $(-\frac{1}{\sqrt{3}},-\frac{14}{9})$ are inflection points.

(d) Since $f(-x)=f(x)$, the graph is symmetric about the $y$-axis. From the previous question, we know that the $x$-intercepts are (approximately) $\pm 1.553$.




23. Use the first and second derivative to sketch the graph of $y=3x^5-5x^3$. Identify all relative extremum and inflection points.

*Answer:*




Relative max at $(-1,2)$, relative min at $(1,-2)$, inflection points at $(0,0)$, $(\frac{\sqrt{2}}{2},\frac{-7\sqrt{2}}{8})$, and $(\frac{-\sqrt{2}}{2},\frac{7\sqrt{2}}{8})$.

24. An open-top box is to be made by cutting away four congruent squares of side length $x$ from the corners of a 50 cm $\times$ 40 cm piece of cardboard. Find the value of $x$ that gives a box of maximum volume.

*Answer:* $x\approx 7.36$ cm

25. A peanuts manufacturer wishes to design a can to hold dry-roasted peanuts. The volume of the cylindrical can is $1000\,\text{cm}^3$, and the circular top of the can is made from aluminum while the sides and bottom are made from stainless steel. If aluminum is twice as expensive as stainless steel, what are the most economical dimensions of the can?

    *Answer:* The radius is $r = \sqrt[3]{\frac{1000}{3\pi}} \approx 4.73$ cm, and the height is $h = \frac{1000}{\pi r^2} \approx 14.2$ cm.

26. Find the point $(x, y)$ on the curve of equation $y = \sqrt{x}$ that is the closest to point $(1, 0)$.

    *Answer:* $(x, y) = \left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$

27. Use differentials to estimate $\sqrt[3]{8.3}$.

    *Answer:* $\sqrt[3]{8.3} \approx 2 + \frac{0.3}{12} = 2.025$

28. The volume of a sphere of radius $r$ is $V = \frac{4}{3}\pi r^3$. Use differentials to estimate both the absolute error and the relative error of the volume if $r = (1.5 \pm 0.2)\,\text{cm}$.

    *Answer:* Absolute error: $\Delta V \approx 5.7\,\text{cm}^3$. Relative error: $\frac{\Delta V}{V} \approx 40\%$.

29. Find and simplify the derivative of the following functions.

    (a) $y = x^3 \cos(x^2 + 1)$ (b) $y = e^{-2x} \sin(3x) + \ln(\cos(2x))$

    (c) $y = \sec^2(3x) + \tan(4x)$ (d) $y = \sin(\pi x) + \cos^3(5x^2)$

    (e) $y = \ln \sqrt[3]{\dfrac{x}{x^3 + 1}}$ (f) $y = 2x \tan^{-1} x - \ln(1 + x^2)$

    *Answers:*

    (a) $y' = 3x^2 \cos(x^2 + 1) - 2x^4 \sin(x^2 + 1)$

    (b) $y' = -2e^{-2x} \sin(3x) + 3e^{-2x} \cos(3x) - 2\tan(2x)$

    (c) $y' = 6\sec^2(3x) \tan(3x) + 4\sec^2(4x)$

    (d) $y' = \pi \cos(\pi x) - 30x \cos^2(5x^2) \sin(5x^2)$

    (e) $y' = \dfrac{1 - 2x^3}{3x(x^3 + 1)}$

    (f) $y' = 2\tan^{-1} x$

30. Consider the curve of equation:

$$\sin(xy) + e^{-2x} = 2 + y^2 - \cos(2x).$$

    Use implicit differentiation to find the derivative of $y$ with respect to $x$.

    *Answer:* $\dfrac{dy}{dx} = \dfrac{2e^{-2x} + 2\sin(2x) - y\cos(xy)}{x\cos(xy) - 2y}$

31. Consider the curve of equation: $8\tan^{-1}\left(\frac{2x}{y}\right) = \pi x^2 y$. Use implicit differentiation to find the slope of the tangent line at the point $(1, 2)$.

    *Answer:* $\frac{4-4\pi}{2+\pi} \approx -1.67$

32. Find all values of $x$ where the derivative of $y = \ln(4x^2 + 1) - 2x^2$ is zero. Test if the points correspond to relative maximum or minimum

*Answer:* $x = 0 \to$ relative min, $x = \pm 1/2 \to$ relative max.

33. Evaluate

$$\int_0^1 x^3(2x^4 + 1)^5 \, dx$$

*Answer:* $\frac{91}{6}$

34. Evaluate

$$\int \sqrt{x}\,(x+2)\,dx$$

*Answer:* $\frac{2}{5}x^{5/2} + \frac{4}{3}x^{3/2} + C$

35. Evaluate

$$\int \frac{x^2}{(2x^3+1)^5}\,dx$$

*Answer:* $\frac{-1}{24(2x^3+1)^4} + C$

36. Find $y$ in terms of $x$ if $\frac{dy}{dx} = \sqrt{6x-3}$ and the curve $y = f(x)$ passes through point $(2, -1)$.

*Answer:* $y = \frac{1}{9}(6x-3)^{3/2} - 4$

37. (a) Approximate

$$\int_0^1 \sqrt{1+x^3}\,dx$$

by using the trapezoidal rule with $n = 5$.

*Answer:* $\approx 1.115$

(b) Use Simpson's rule to approximate

$$\int_{1.0}^{2.8} f(x)\,dx$$

using the following data points.

| $x$ | 1.0 | 1.3 | 1.6 | 1.9 | 2.2 | 2.5 | 2.8 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 3.2 | 4.1 | 5.2 | 4.6 | 4.2 | 5.1 | 5.7 |

*Answer:* $\approx 8.29$

38. An object starts from an initial position of 5 m, with an initial velocity of 10 m/s (at time $t = 0$). Determine its position $x$ as a function of time $t$ if its acceleration (in m/s$^2$) is given by

$$a = 12 - 0.6t$$

*Answer:* $x = 6t^2 - \frac{t^3}{10} + 10t + 5$

39. In coming to a stop, the acceleration of a car is $a = -5t$. If it is travelling at 40 m/s when the brakes are applied, how far does it travel while stopping ?

    *Answer:* 106.7 m

40. As a rocket burns, it consumes fuel and consequently gets lighter in mass. If a Saturn V rocket initially starts out with $2 \times 10^6$ kg of fuel and the rate of change of the mass of fuel (in kg/s) is given by

$$\frac{dm}{dt} = -t\sqrt{t^2 + 100},$$

    how long does it take to burn all the fuel?

    *Answer:* 181.4 s

41. Find the area bounded by the parabola $y = x^2$, and the line $y = x + 2$.

    *Answer:* $A = \frac{9}{2}$

42. Find the area bounded by the curves $y = x^3 - 3$, $x = 2$, $y = -1$, and $y = 3$.

    *Answer:* $A \approx 1.7128$

43. Find the volume of the solid generated by revolving around the $x$-axis the first-quadrant area bounded by $y = 9 - x^2$, $x = 0$, and $y = 0$.

    Answer: $V = \frac{648\pi}{5} \approx 407.15$

44. Use the shell method to find the volume of the solid generated by revolving around the $y$-axis the area bounded by $y = 3x^2 - x^3$ and $y = 0$.

    Answer: $V = \frac{243\pi}{10} \approx 76.34$

45. Find the coordinates of the centroid of the area bounded by the line $y = x$, and the parabola $y = x^2$.

    Answer: $(\overline{x}, \overline{y}) = \left(\frac{1}{2}, \frac{2}{5}\right)$

46. Find the coordinates of the centroid of the area bounded by $y = x^3$, $y = 0$, and $x = 2$.

    Answer: $(\overline{x}, \overline{y}) = \left(\frac{8}{5}, \frac{16}{7}\right)$

47. Consider the following matrices.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 4 \\ 1 & 0 \\ 2 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

    Evaluate: (a) $2AC + 3D$ (b) $BC$ (c) $D^{-1}$.

    *Answers:*

    (a) $\begin{bmatrix} 20 & 15 \\ 23 & 36 \end{bmatrix}$

    (b) $\begin{bmatrix} 13 & 12 \\ 7 & 8 \\ 13 & 14 \end{bmatrix}$

(c) $\begin{bmatrix} 4/5 & -1/5 \\ -3/5 & 2/5 \end{bmatrix}$

48. Solve the following systems of linear equations.

(a)

$$\begin{cases} x + 2y - z = 2 \\ 3x + 7y - 5z = 5 \\ -x - 2y = 1 \end{cases}$$

(b)

$$\begin{cases} x + y + z = 2 \\ x - 4y - z = 3 \\ 2x + 3y + 5z = 9 \end{cases}$$

*Answers:*

(a) $x = 13,\ y = -7,\ z = -3$

(b) $x = 1,\ y = -1,\ z = 2$