26.6 Applications

$W = F \cdot d$ when force is constant

More generally:

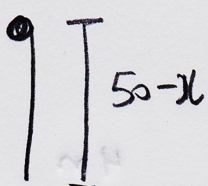
$$W = \int_a^b F(x)\,dx$$

$F(x)$: force applied at distance x

<u>Ex</u>: Find the work done in winding up 30m of a 50m chain that weighs 2 N/m.

Let x = length of chain already wound up.

$$0 \le x \le 30$$



x must be increasing to avoid a sign error

$$F(x) = 2(50 - x) \quad N/m \cdot m$$
$$= 100 - 2x \quad N$$

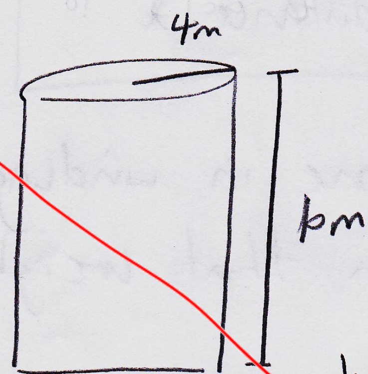
$$W = \int_0^{30} \overset{N}{F(x)}\,\overset{m}{dx}$$

$$= \int_0^{30} (100-2x)\,dx$$

$$= \left[100x - x^2\right]_0^{30}$$

$$= 2100\ N\cdot m \quad \text{or} \quad 2100\ J$$

Ex:



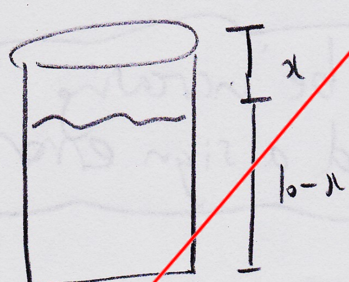
Cylindrical tank, initially full of water.

Water weighs $9800\ N/m^3$.

Work done in pumping out the tank?

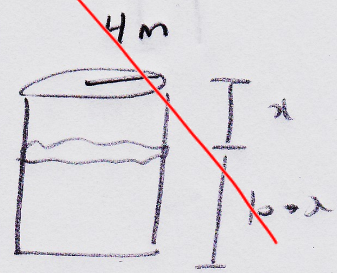
OMIT

Let x = height of water already pumped out



$0 \le x \le 10$

Imagine a dish of water: dx



$V_{dish} = \pi \cdot \text{radius}^2 \cdot \text{thickness}$
$= \pi 4^2\, dx$

Work = ~~Force~~ Weight · distance

$= 9800 \; N/m^3 \cdot$ volume · distance

$dW = 9800 \; (\pi \cdot 4^2 dx) \, x$

$dW = 9800(16)\pi \, x \, dx$

$W = \int_0^{10} 9800(16)\pi \, x \, dx$

$= 9800(16)\pi \left. \frac{x^2}{2} \right|_0^{10}$

$= 2.5 \times 10^7 \; J$

OMIT

__Ex:__ Gravitational attraction between 2 objects is $F = \frac{k}{x^2}$, where x = distance between objects. Find work done in moving the objects from 20m apart to 40m apart.

x = distance between objects

$20 \leq x \leq 40$

$W = \int_{20}^{40} F(x) \, dx$

$$= \int_{20}^{40} \frac{k}{x^2}\, dx$$

$$= -k x^{-1} \Big|_{20}^{40}$$

$$= -0.025k + 0.05k$$

$$= 0.025k \text{ J}$$

<u>Ex</u>: A spring of natural length 1m requires a force of 8N to stretch it to 3m. Find the work done in stretching it to 3.5m.

2.5m of stretch

2m of stretch

$$F = kx$$

$$8 = k(2)$$

$$k = 4 \text{ N/m} \qquad \boxed{F = 4x}$$

$$W = \int F(x)\,dx$$

x = amount of stretch

$0 \le x \le 2.5$

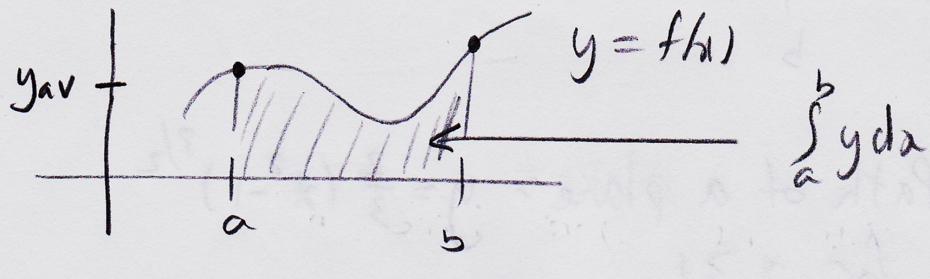
$$= \int_0^{2.5} 4x\,dx$$

$$= \left[2x^2\right]_0^{2.5}$$

$$= 12.5 \text{ J}$$

Average value of a function

$$\boxed{y_{av} = \frac{\int_a^b y\,dx}{b-a}}$$



Ex: Find average value of $y = x^4$ from $x=0$ to $x=2$

$$y_{av} = \frac{\int_0^2 x^4\,dx}{2-0}$$

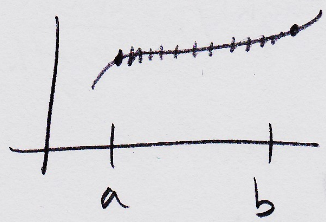
$$= \frac{1}{2}\int_0^2 x^4\,dx$$

$$= \frac{1}{2}\left[\frac{x^5}{5}\right]_0^2$$

$$= \frac{1}{2}\left(\frac{32}{5}\right)$$

$$= \frac{16}{5}$$

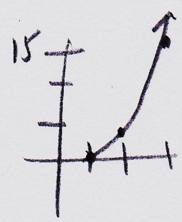
Arc Length



$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Ex: Path of a plane: $y = \frac{2}{3}(x^2-1)^{3/2}$ for $x \geq 1$.

Find length of path for $1 \leq x \leq 3$



$$\frac{dy}{dx} = \frac{2}{3} \cdot \frac{3}{2}(x^2-1)^{1/2}(2x)$$

$$= \sqrt{x^2-1}\,(2x)$$

$$\left(\frac{dy}{dx}\right)^2 = (x^2-1)(4x^2)$$

$$= 4x^4 - 4x^2$$

$$s = \int_1^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_1^3 \sqrt{4x^4 - 4x^2 + 1}\, dx$$

$$= \int_1^3 \sqrt{(2x^2-1)^2}\, dx$$

$\sqrt{\#^2} = \#$ when $\# > 0$

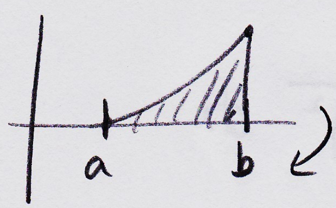
$$= \int_1^3 (2x^2 - 1)\, dx$$

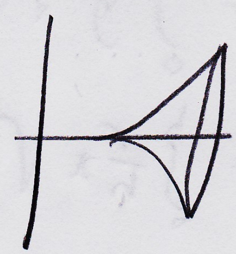
$$= \left[\frac{2x^3}{3} - x\right]_1^3$$

$$= \left(\frac{54}{3} - 3\right) - \left(\frac{2}{3} - 1\right)$$

$$= \frac{52}{3} - 2$$

$$= \frac{46}{3}$$





Surface area of Solid of Revolution

$$S = 2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

<u>Ex</u>: Find surface area of a cone with height h and radius r.

Line $y = mx + b$

$y = \frac{r}{h}x + 0$

$$\boxed{y = \frac{r}{h}x}$$

$$\frac{dy}{dx} = \frac{r}{h}$$

$$S = 2\pi \int_0^h y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_0^h \frac{r}{h}x \sqrt{1 + \left(\frac{r}{h}\right)^2}\, dx$$

$$= 2\pi \frac{r}{h} \sqrt{1 + \left(\frac{r}{h}\right)^2} \int_0^h x\, dx$$

$$= 2\pi \frac{r}{h} \sqrt{1 + \left(\frac{r}{h}\right)^2} \left[\frac{x^2}{2}\right]_0^h$$

$$= 2\pi \frac{r}{h} \sqrt{1 + \frac{r^2}{h^2}} \left(\frac{h^2}{2}\right)$$

$$= \pi r h \sqrt{1 + \frac{r^2}{h^2}}$$

$\sqrt{h^2}$

$$= \pi r \sqrt{h^2 + r^2}$$