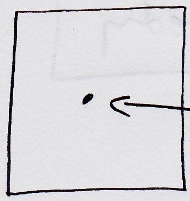
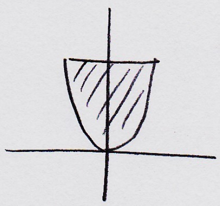
26.4 Centroids

Thin metal plate of constant density and thickness



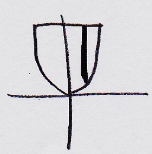
centre of mass
"centroid"
$(\bar{x}, \bar{y})$



Centroid ?

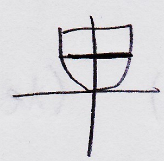
Can use vertical or horizontal slices
Let (x_e, y_e) = centroid of slice

Vertical Slices



$$(x_e, y_e) = \left(x, \frac{y_b + y_t}{2}\right)$$

Horizontal Slices

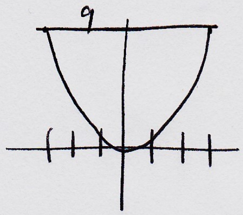


$$(x_e, y_e) = \left(\frac{x_l + x_r}{2}, y\right)$$

$$\bar{x} = \frac{1}{A} \int_A x_e \, dA \qquad \bar{y} = \frac{1}{A} \int_A y_e \, dA$$

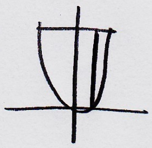
integrate over the entire region

Ex: Region bounded by $y = x^2$ and $y = 9$.
Find the centroid.



$\boxed{\bar{x} = 0 \text{ by symmetry}}$ ☺

Find $\bar{y}$:



1) Find area A

$$A = \int \boxed{(y_t - y_b)\,dx} \leftarrow dA$$

$$= \int_{-3}^{3} (9 - x^2)\,dx$$

$$= \left[9x - \frac{x^3}{3} \right]_{-3}^{3}$$

$$= (27 - 9) - (-27 + 9)$$

$$= 36$$

2) (x_e, y_e)

Vertical Slice $(x_e, y_e) = \left(x, \frac{y_t + y_b}{2}\right)$

$$= \left(x, \frac{9 + x^2}{2}\right)$$

3) $\int_A y_e \, dA$

$$= \int_{-3}^{3} \left(\frac{9+x^2}{2}\right) \underbrace{(9-x^2)\,dx}_{dA} \quad \leftarrow y_e$$

$$= \frac{1}{2} \int_{-3}^{3} (9+x^2)(9-x^2)\,dx$$

$$= \frac{1}{2} \int_{-3}^{3} (81 - x^4)\,dx$$

$$= \frac{1}{2} \left[81x - \frac{x^5}{5}\right]_{-3}^{3}$$

$$= \frac{1}{2}\left[\left(243 - \frac{243}{5}\right) - \left(-243 + \frac{243}{5}\right)\right]$$

$$= \frac{972}{5}$$

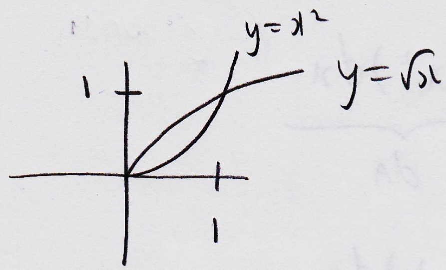
4) $\bar{y} = \frac{1}{A} \int_A y_e \, dA$

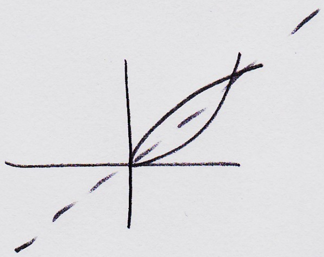
$$= \frac{1}{36} \cdot \frac{972}{5}$$

$$= \frac{27}{5}$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{27}{5}\right) \checkmark$$

Ex: Plane region bounded by $y=\sqrt{x}$ and $y=x^2$. Centroid?





Region is symmetric about the line $y=x$

Means $\bar{y}=\bar{x}$ ☺

→ Find $\bar{x}$

1) Find area A

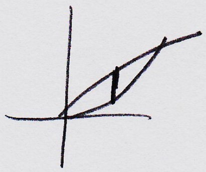
$$A = \int \boxed{(y_t - y_b)\,dx} \leftarrow dA$$

$$= \int_0^1 (\sqrt{x} - x^2)\,dx$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{3}$$

2) (x_e, y_e)



Vertical Slice $(x_e, y_e) = \left(x, \frac{y_t + y_b}{2}\right)$

$= \left(x, \frac{\sqrt{x} + x^2}{2}\right)$

3) $\int_A x_e \, dA$

$$= \int_0^1 \overbrace{x}^{x_e} \underbrace{(\sqrt{x} - x^2)\,dx}_{dA}$$

$$= \int_0^1 (x^{3/2} - x^3)\,dx$$

$$= \left[\frac{2}{5}x^{5/2} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{2}{5} - \frac{1}{4}$$

$$= \frac{3}{20}$$

4) $\bar{x} = \frac{1}{A} \int_A x_e \, dA$

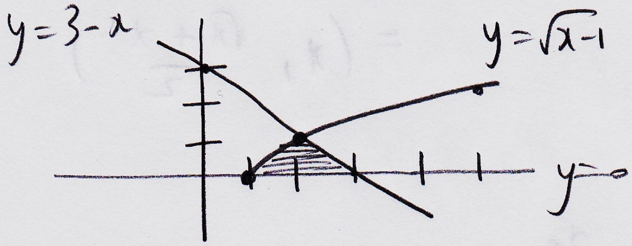
$$= 3 \cdot \frac{3}{20}$$

$$= \frac{9}{20}$$

$$(\bar{x}, \bar{y}) = \left(\frac{9}{20}, \frac{9}{20}\right)$$ ✓

Ex: Region bounded by $y = 3 - x$, $y = \sqrt{x-1}$ and $y = 0$.

Find $\bar{y}$.



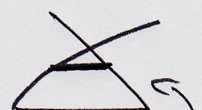
$y = \sqrt{x-1}$

x	y
1	0
2	1
5	2

No symmetry ☹

1) Find area A

Horizontal slices



$y = 3 - x$

$\boxed{x_r = 3 - y}$

$y = \sqrt{x-1}$

$y^2 = x - 1$

$y^2 + 1 = x$

$\boxed{x_l = y^2 + 1}$

$$A = \int_0^1 \boxed{(x_r - x_l)\,dy}$$

$$= \int_0^1 \left[(3-y) - (y^2+1)\right] dy$$

$$= \int_0^1 \underbrace{(2 - y - y^2)\,dy}_{dA}$$

$$= \left[2y - \frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$$

$$= \frac{7}{6}$$

2) (x_e, y_e)

Horizontal slice

$$(x_e, y_e) = \left(\frac{x_l + x_r}{2}, y\right)$$

$$= \left(\frac{y^2+1+3-y}{2}, y\right)$$

$$= \left(\frac{y^2-y+4}{2}, y\right)$$

3) $\int_A y_e \, dA$

$$= \int_0^1 \overset{y_e}{y} \underbrace{(2-y-y^2)\,dy}_{dA}$$

$$= \int_0^1 (2y - y^2 - y^3)\,dy$$

$$= \left[y^2 - \frac{y^3}{3} - \frac{y^4}{4}\right]_0^1$$

$$= \frac{5}{12}$$

4) $\bar{y} = \frac{1}{A}\int_A y_e \, dA$

$$= \frac{6}{7} \cdot \frac{5}{12}$$

$$= \frac{5}{14}$$