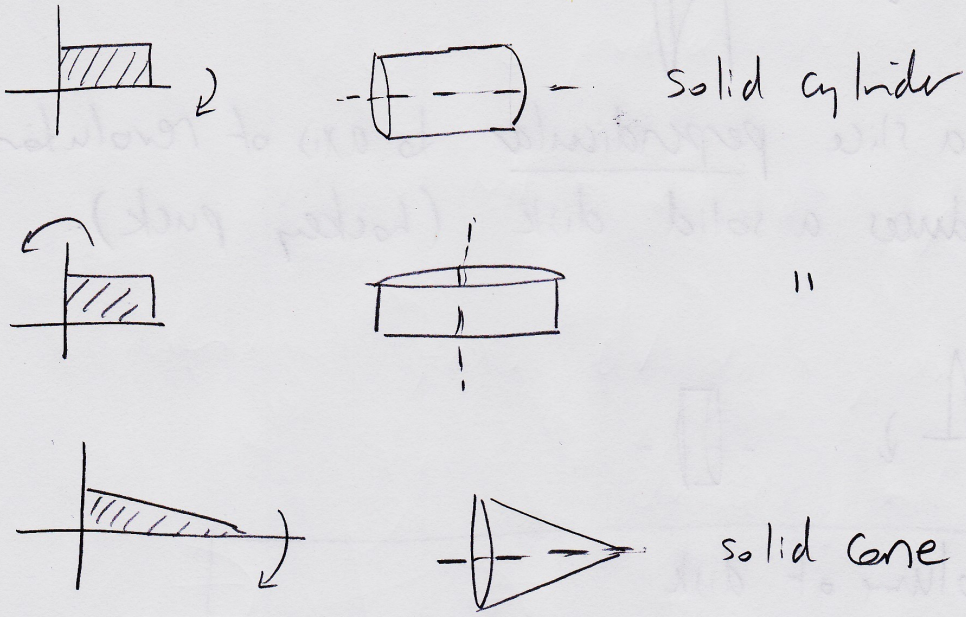
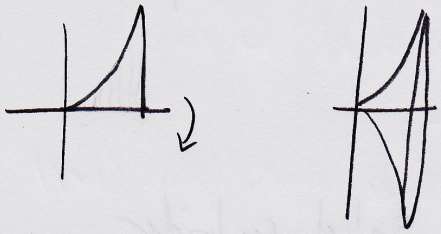
26.3 Volumes by Integration

Revolve an area around an axis to produce a solid.

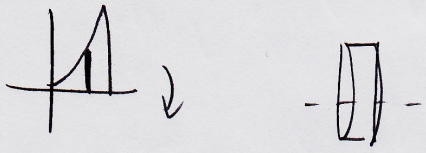


Goal: Calculate volumes

Disk Method



Take a slice <u>perpendicular</u> to axis of revolution
Produces a solid disk (hockey puck).

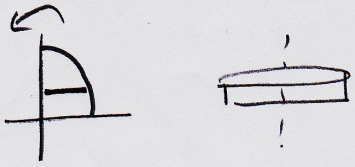


Volume of disk

$$dV = \pi \cdot \text{radius}^2 \cdot \text{thickness}$$

About x-axis: $dV = \pi y^2 dx$

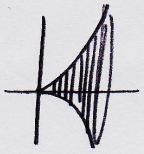
About y-axis



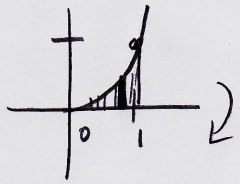
$dV = \pi x^2 dy$

Total volume = sum of dV

$V = \int dV$



Ex: Revolve region bounded by $y = x^2$, $y = 0$, $x = 0$, $x = 1$ about x-axis.
Volume?



$dV = \pi \cdot \text{radius}^2 \cdot \text{thickness}$

$= \pi y^2 \, dx$

$$V = \int_0^1 \pi y^2 \, dx$$

$$= \pi \int_0^1 (x^2)^2 \, dx \qquad \leftarrow y^2$$

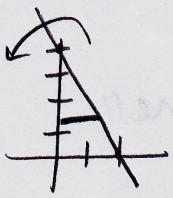
$$= \pi \int_0^1 x^4 \, dx$$

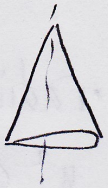
$$= \pi \left[\frac{x^5}{5} \right]_0^1$$

$$= \frac{\pi}{5}$$

Ex: Revolve first-quadrant region bounded by $y = 4 - 2x$ above y-axis.

Volume?





$dV = \pi \cdot \text{radius}^2 \cdot \text{thickness}$

$= \pi x^2 \, dy$

$$V = \int_0^4 \pi x^2 \, dy$$

$2x = 4 - y$

$x = \frac{4-y}{2}$

$x^2 = \frac{16 - 8y + y^2}{4}$

$x^2 = 4 - 2y + \frac{y^2}{4}$

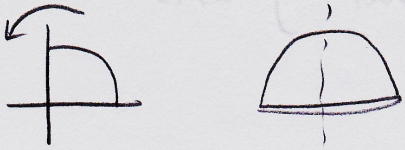
$$= \pi \int_0^4 \left(4 - 2y + \frac{y^2}{4}\right) dy$$

$$= \pi \left[4y - y^2 + \frac{y^3}{12}\right]_0^4$$

$$= \pi \left[\frac{64}{12} - 0\right]$$

$$= \frac{16\pi}{3}$$

Shell Method



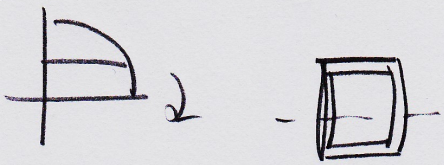
Take a slice <u>parallel</u> to axis of revolution

Produces a hollow shell

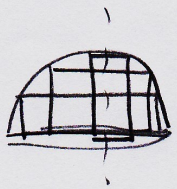
Volume of shell: $dV = 2\pi \cdot \text{radius} \cdot \text{height} \cdot \text{thickness}$

About y-axis: $dV = 2\pi x y\, dx$

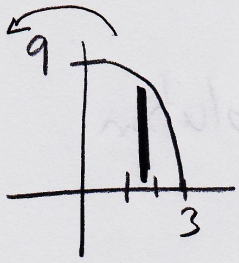
About x-axis: $dV = 2\pi y x\, dy$



$V = \int dV$



Ex: Revolve first-quadrant region bounded by $y = 9 - x^2$ about y-axis. Find volume using shells.



$dV = 2\pi \cdot \text{radius} \cdot \text{height} \cdot \text{thickness}$

$dV = 2\pi x y \, dx$

$$V = \int_0^3 2\pi x y \, dx$$

$$= 2\pi \int_0^3 x(9 - x^2) \, dx \quad \leftarrow y$$

$$= 2\pi \int_0^3 (9x - x^3) \, dx$$

$$= 2\pi \left[\frac{9x^2}{2} - \frac{x^4}{4} \right]_0^3$$

$$= 2\pi \left[\left(\frac{81}{2} - \frac{81}{4} \right) - 0 \right]$$

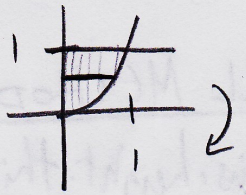
$$= \frac{2\pi \cdot 81}{4}$$

$$= \frac{81\pi}{2}$$

first-quadrant

Ex: Revolve region bounded by $y = x^2$, $y = 1$, $x = 0$ about x-axis.

Volume using shells?



$dV = 2\pi \cdot \text{radius} \cdot \text{height} \cdot \text{thickness}$

$dV = 2\pi y\, x\, dy$

$$V = \int_0^1 2\pi y x \, dy$$

$y = x^2$

$x^2 = y$

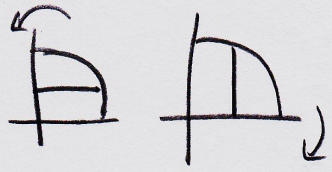
$x = \pm\sqrt{y}$

$x = -\sqrt{y}$ $\quad$ $x = \sqrt{y}$

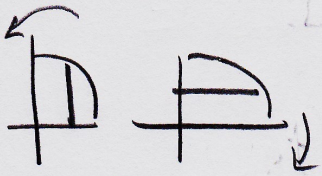
$x = \sqrt{y}$

$$= 2\pi \int_0^1 y\sqrt{y}\, dy$$

$$= 2\pi \int_0^1 y^{3/2}\, dy = 2\pi \left[\tfrac{2}{5} y^{5/2}\right]_0^1 = \frac{4\pi}{5}$$



DISK $dV = \pi \cdot \text{radius}^2 \cdot \text{thickness}$



SHELL $dV = 2\pi \cdot \text{radius} \cdot \text{height} \cdot \text{thickness}$