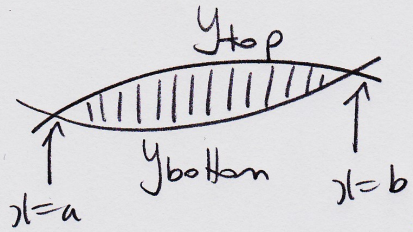
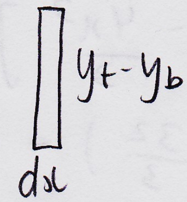
26.2 Area

Area Between Curves





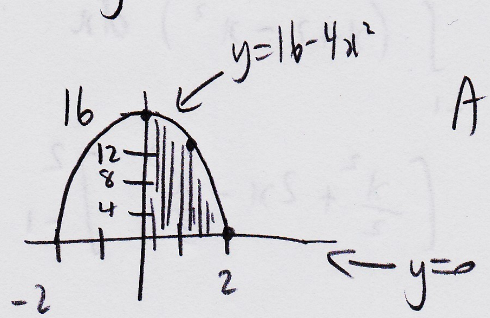
Vertical slices or "vertical elements"
have area $(y_t - y_b)dx$ ⟵ call this dA

A = sum of infinitely-many slices

$A = \int dA$

$$\boxed{A = \int_a^b (y_t - y_b)\,dx}$$

<u>Ex</u>: Find first-quadrant area bounded by
$y = 16 - 4x^2$



$$A = \int_a^b (y_t - y_b)\,dx$$

$$= \int_0^2 (16 - 4x^2 - 0)\, dx$$

$$= \left[16x - \frac{4x^3}{3} \right]_0^2$$

$$= \left(32 - \frac{32}{3}\right) - 0$$

$$= \frac{64}{3}$$

<u>EX</u>: Area between $y = x^2$ and $y = x + 2$?

Intersection?

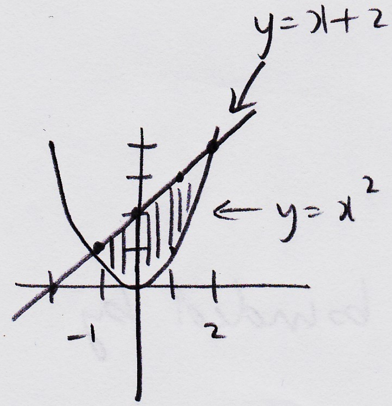
$$y = y$$

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = 2, -1$$



$$A = \int_a^b (y_t - y_b)\, dx$$

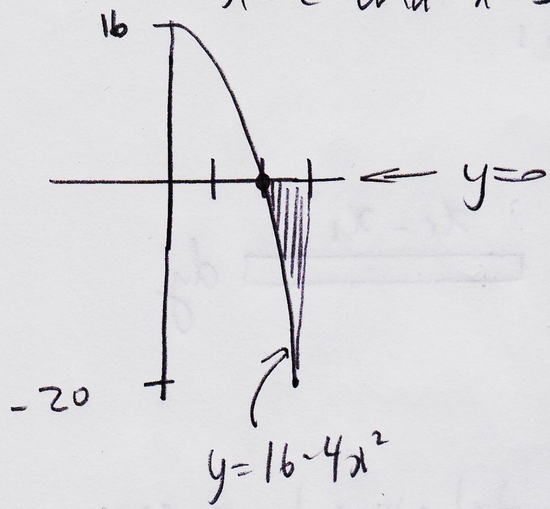
$$= \int_{-1}^{2} (x + 2 - x^2)\, dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2}$$

$$= \left(6 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$$

$$= 9/2$$

<u>Ex</u>: Area bounded by $y = 16 - 4x^2$, $y = 0$, $x = 2$ and $x = 3$?



$$A = \int_a^b (y_t - y_b)\,dx$$

$$= \int_2^3 \left[0 - (16 - 4x^2)\right] dx$$

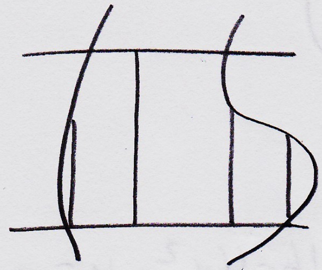
$$= \int_2^3 (-16 + 4x^2)\,dx$$

$$= \left[-16x + \frac{4x^3}{3}\right]_2^3$$

$$= -12 - \left(-32 + \frac{32}{3}\right)$$

$$= 20 - 32/3$$

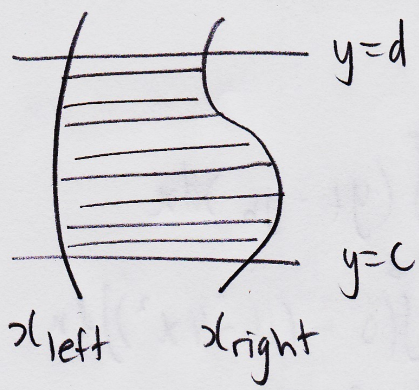
$$= 28/3$$

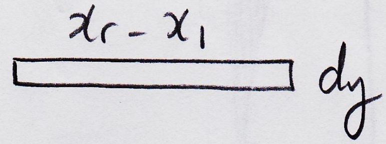


Inconvenient to take vertical slices

y_t and y_b vary

Take horizontal slices



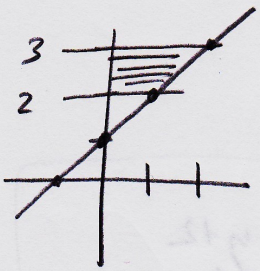


Horizontal slices have area

$$dA = (x_r - x_\ell)\,dy$$

$$\boxed{A = \int_c^d (x_r - x_\ell)\,dy}$$

Ex: Area bounded by $y = x+1$, $x = 0$, $y = 2$ and $y = 3$?

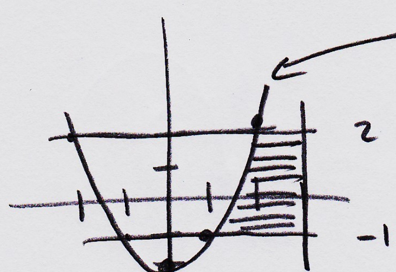


$y = x+1$

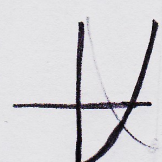
$$\boxed{\begin{aligned} y &= x+1 \\ y-1 &= x \\ x_r &= y-1 \end{aligned}}$$

$$\begin{aligned} A &= \int_c^d (x_r - x_l)\,dy \\ &= \int_2^3 (y-1-0)\,dy \\ &= \left[\frac{y^2}{2} - y\right]_2^3 \\ &= \left(\frac{9}{2} - 3\right) - (2-2) \\ &= 3/2 \end{aligned}$$

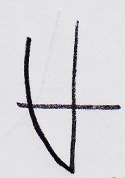
Ex: Area bounded by $y = x^2 - 2$, $x = 3$, $y = -1$ and $y = 2$



$$\begin{aligned} y &= x^2 - 2 \\ y+2 &= x^2 \\ x^2 &= y+2 \\ x &= \pm\sqrt{y+2} \end{aligned}$$



$x = \sqrt{y+2}$



$x = -\sqrt{y+2}$

$$\boxed{x_l = \sqrt{y+2}}$$

$$A = \int_c^d (x_r - x_l)\,dy$$

$$= \int_{-1}^{2} \left(3 - \sqrt{y+2}\right)dy$$

Sub $u = y+2$

$du = dy$

when $y = -1$, $u = 1$

$y = 2$, $u = 4$

$$= \int_1^4 (3 - \sqrt{u})\,du$$

$$= \left[3u - \frac{2}{3}u^{3/2}\right]_1^4$$

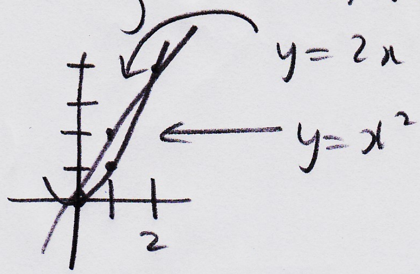
$$= \left(12 - \frac{16}{3}\right) - \left(3 - \frac{2}{3}\right)$$

$$= 9 - 14/3$$

$$= 13/3$$

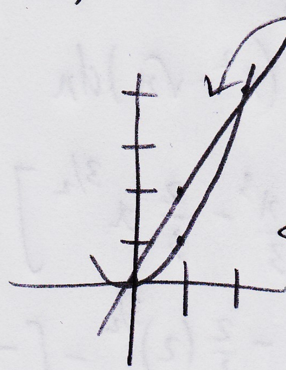
Ex: Set up area bounded by $y = x^2$ and $y = 2x$

using: a) vertical slices



$$A = \int_0^2 (2x - x^2)\,dx$$

b) horizontal slices



$y = 2x$

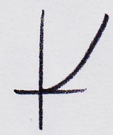
$x = y/2$

$x_l = y/2$

$y = x^2$

$x^2 = y$

$x = \pm\sqrt{y}$



$x = \sqrt{y}$

$x = -\sqrt{y}$

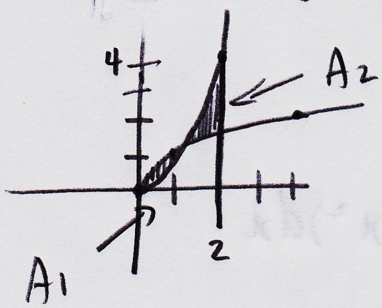
$x_r = \sqrt{y}$

$$A = \int_c^d (x_r - x_l)\,dy$$

$$= \int_0^4 (\sqrt{y} - y/2)\,dy$$

$\boxed{\text{Answer: } \frac{4}{3}}$

Ex: Total area bounded by $y = x^2$, $y = \sqrt{x}$, $x = 0$ and $x = 2$?



$$A = A_1 + A_2$$

A_1

$$A_1 = \int_0^1 (\sqrt{x} - x^2)\,dx$$
$$= \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1$$
$$= \frac{1}{3}$$

A_2

$$A_2 = \int_1^2 (x^2 - \sqrt{x})\,dx$$
$$= \left[\frac{x^3}{3} - \frac{2}{3}x^{3/2}\right]_1^2$$
$$= \frac{8}{3} - \frac{2}{3}(2)^{3/2} - \left[-\frac{1}{3}\right]$$
$$= 3 - \frac{2}{3}(2)^{3/2}$$

$$A = A_1 + A_2$$
$$= \frac{10}{3} - \frac{2}{3}(2)^{3/2}$$