24.8 Differentials and Linear Approximation

Ex: $y = 3x^6 + 7x^3$

$$\frac{dy}{dx} = 18x^5 + 21x^2$$

$$dy = (18x^5 + 21x^2)\,dx$$

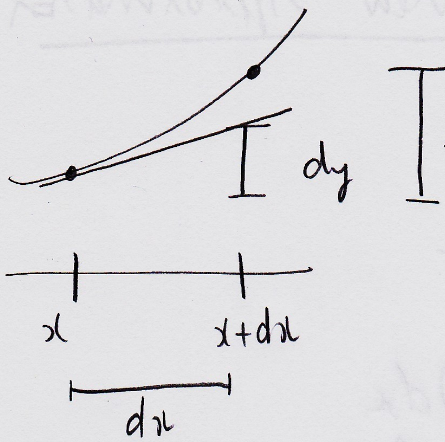
differential of y — differential of x

Ex: Find the differential of V

$$V = \frac{4}{3}\pi r^3$$

$$dV = 4\pi r^2\,dr$$

Application: Error calculation



$\Delta y = f(x+dx) - f(x)$

Δx or dx : small change in x

dy : rise of tangent line

Δy : change in y

When dx is small, $\Delta y \simeq dy$

<u>Caution</u>: Say $y = x^2$

x is the independent variable

Δx and dx mean the same thing

y is the dependent variable

Δy and dy are different

<u>Ex</u>: $f(x) = x^2 + 2$

Compute Δy and dy for $x=1$ and $dx = 0.05$

$$dy = f'(x)\,dx$$
$$dy = 2x\,dx$$
$$dy = 2(1)(0.05) = 0.1$$

approx. change in y

$$\Delta y = f(x+dx) - f(x)$$
$$= f(1.05) - f(1)$$
$$= (1.05^2 + 2) - 3$$
$$= 0.1025$$

true change in y

<u>Ex</u>: Approximate Δy using dy

$y = x^{1/2}$ $\quad x = 4 \quad dx = 0.1$

$$\Delta y \approx dy$$
$$= \tfrac{1}{2} x^{-1/2}\,dx$$
$$= \tfrac{1}{2}(4)^{-1/2}(0.1)$$
$$= 0.025$$

Ex: Sphere's radius is measured to be 2.500 cm. If the true radius is 2.512 cm, estimate the error in volume.

$r = 2.5$ (measured)

dr = error in radius
= true − measured
= 0.012

$$V = \frac{4}{3}\pi r^3$$

$$dV = 4\pi r^2 dr$$

error in volume

$$= 4\pi(2.5)^2(0.012)$$

$$\approx 0.9425 \text{ cm}^3$$

Why use differentials?

Absolute error in radius : dr

Relative error in radius : $\frac{dr}{r}$

Relative error in volume : $\frac{dV}{V} = \frac{4\pi r^2 dr}{\left(\frac{4}{3}\pi r^3\right)}$

$= 3 \frac{dr}{r}$:)

Comes from dimension of sphere

No predictable relationship between

$\frac{\Delta V}{V}$ and $\frac{\Delta r}{r}$:(

Ex: Given $A = \pi r^2$ $\frac{dr}{r} = 0.02$

Find $\frac{dA}{A}$

$$\frac{dA}{A} = \frac{2\pi r\,dr}{\pi r^2}$$
$$= \frac{2\,dr}{r}$$
$$= 2(0.02)$$
$$= 0.04$$

2% relative error in radius of circle

$\Rightarrow$ 4% " area

Ex: Given $F = \frac{k}{r^2}$, k constant

Show $\frac{dF}{F} = -2\frac{dr}{r}$

OMIT

$\rightarrow F = kr^{-2}$

$$\frac{dF}{F} = \frac{-2kr^{-3}dr}{kr^{-2}}$$
$$= \frac{-2\,dr}{r}$$
$$= -2\,\frac{dr}{r}$$

Let $a = \#$ near x such that $f(a)$ is known

$f(x) = f(a) + \text{error}$

$$\boxed{f(x) \approx f(a) + f'(a)(x-a)}$$

small change in x

"linearization of $f(x)$ at $x=a$"

Valid near $x=a$

Ex: a) Find the linearization of $f(x) = x^4$ at $x = 2$

$\boxed{a=2}$

$f(x) \approx f(2) + f'(2)(x-2)$ $\qquad f'(x) = 4x^3$

$$\boxed{f(x) \approx 16 + 32(x-2)}$$ (Valid near $x=2$)

b) Approximate 1.98^4 using part a)

$\boxed{x = 1.98}$

$1.98^4 \approx 16 + 32(1.98-2)$ $\qquad$ easy to calculate

≈ 15.36

Ex: Estimate $\sqrt[3]{8.3}$ using a linearization (a.k.a. linear approximation)

$$\boxed{f(x) = x^{1/3}}$$

a? a: near 8.3 and f(a) known

$$\boxed{a = 8}$$

$$f(x) \approx f(a) + f'(a)(x-a)$$

$$f(x) \approx f(8) + f'(8)(x-8) \qquad f'(x) = \frac{1}{3}x^{-2/3}$$

$$\boxed{f(x) \approx 2 + \frac{1}{12}(x-8)} \qquad f'(8) = \frac{1}{3}(8)^{-2/3} = \frac{1}{12}$$

(Valid near $x=8$)

$$f(8.3) \approx 2 + \frac{1}{12}(8.3-8)$$

$$= 2.025$$

Application of Linearization (beyond Math 191)

m_{tan}, rate of change

$f(x) = x^4$ ↑ area, volume etc.

$f'(x) = 4x^3$

$f'(x) = 4x^3 \Rightarrow f(x) = x^4 + C$ easy ish

$f'(x) = e^{x^2} \Rightarrow f(x) = ?$ hard

Use linearization to simplify $f'(x)$