MAXIMIZING OR MINIMIZING A FUNCTION

Example 1. Find two nonnegative numbers that sum to 50 and whose product is maximized.

Example 2. Cut the corners from a 15cm $\times$ 15cm metal sheet to form an open-topped box. Find the height of the box that maximizes its volume. What is the maximum volume?

~~Example 3. An animal pen is built with five pieces of fencing: four pieces enclosing a rectangular area and a fifth piece dividing the pen in two. Given 400m of fencing, what is the maximum area that can be enclosed? What are the dimensions of the pen?~~ OMIT

Example 4. Find the point on the line $y = 2x + 3$ that is closest to the point $(2, 1)$.

Example 5. A closed rectangular box has a square base. The material for the top and sides of the box costs \$1/cm^2. The material for the base costs \$2/cm^2. Find the dimensions that maximize the volume of the box if the total cost must be \$144.

Example 6. A cylinder has volume 100cm^3. What dimensions minimize its surface area?

24.7 Maximizing or Minimizing a Function

① Let x = 1st number

y = 2nd number

1) Maximize $f = xy$

Restriction : $x + y = 50$

2) $\boxed{\text{Use restriction to get } f \text{ in terms of one variable}}$

$y = 50 - x \rightarrow f$

$f(x) = x(50 - x)$

or $f(x) = 50x - x^2$

3) Set $f'(x) = 0$

$f'(x) = 50 - 2x$

$50 - 2x = 0$

$x = 25$

Note: To check that $x = 25$ is a maximum:

"2nd derivative test"

$f''(x) = -2$

$f''(25) < 0$

$\cap$ $f(x)$ is CD at $x = 25$

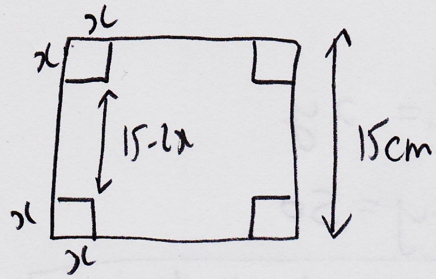
max ✓

4) Answer

$x = 25$

$y = 50 - x = 25$

(2)



1) Maximize $V = lwh$

$V = (15-2x)^2 x$

2) Single variable ✓

3) Set $V' = 0$

$V' = (15-2x)^2 + x \cdot 2(15-2x)(-2)$

$(15-2x)^2 - 4x(15-2x) = 0$

$(15-2x)(15-2x-4x) = 0$

$15-2x = 0$

$x = 7.5$

nonsense

$l = w = 0$

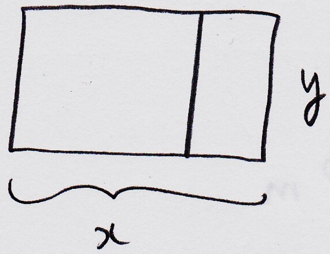
$15-6x = 0$

$x = \frac{15}{6} = 2.5$

4) Answer $x = 2.5$ cm

Max volume is $(15-2x)^2 x = 250\text{cm}^3$

③



OMIT #3

1) Maximize : $A = xy$

Restriction : $2x + 3y = 400$

2) Single variable

$$3y = 400 - 2x$$

$$y = \frac{400 - 2x}{3}$$

$$A = x \cdot \frac{(400 - 2x)}{3}$$

3) set $A' = 0$

$$A' = \frac{400 - 2x}{3} + x\left(-\frac{2}{3}\right)$$

$$\frac{400 - 2x - 2x}{3} = 0$$

$$400 - 4x = 0$$

$$x = 100$$

4) Answer

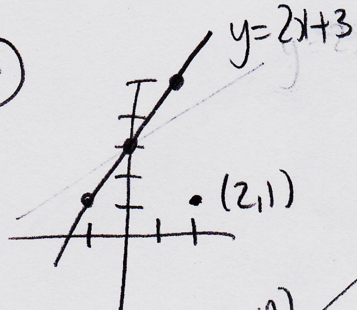
$x = 100\text{ m}$

$y = \dfrac{400-2x}{3} = \dfrac{200}{3}\text{ m}$

OMIT #3

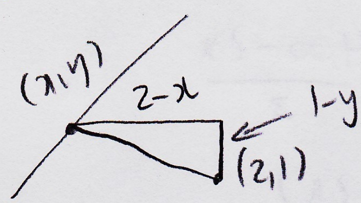
$\text{Area} = \dfrac{20{,}000}{3}\text{ m}^2$

4



Let the point be (x, y)

1)



Minimize $d = \sqrt{(2-x)^2 + (1-y)^2}$

Equivalent: minimize $d^2 = (2-x)^2 + (1-y)^2$
(gives same point (x,y))

Restriction: point on line $y = 2x+3$

2) Single variable

$f = d^2 \big|_{y=2x+3}$

$f = (2-x)^2 + (1-(2x+3))^2$

$f = (2-x)^2 + (-2-2x)^2$

3) Set $f' = 0$

$$2(2-x)(-1) + 2(-2-2x)(-2) = 0$$

$$-4 + 2x + 8 + 8x = 0$$

$$4 + 10x = 0$$

$$x = -0.4$$

4) Answer

$x = -0.4$

$y = 2x+3 = 2.2$ Point $(-0.4, 2.2)$

Note: minimum distance is

$$d = \sqrt{(2-x)^2 + (1-y)^2} \approx 2.7 \text{ units}$$

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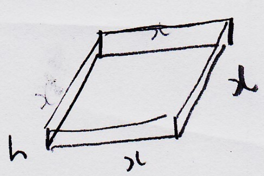


1) Maximize $V = x^2 h$

Restriction: $144 = 1x^2 + 4 \cdot 1 \cdot xh + 2x^2$ $\$ = \$/cm^2 \cdot cm^2$

top 4 sides bottom

2) Single variable

$$h = \frac{144 - 3x^2}{4x}$$

$$V = \frac{\overset{x}{\cancel{x^2}} \cdot (144-3x^2)}{4\cancel{x}}$$

$$V = \frac{144x - 3x^3}{4}$$

3) $V' = 0$

$$\frac{144-9x^2}{4} = 0$$

$$144 - 9x^2 = 0$$

$$x^2 = 16$$

$$x = \pm 4$$

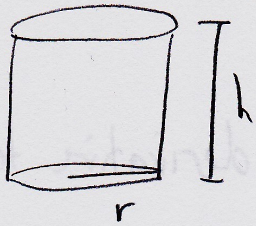
$x = 4$ (length >0)

4) Answer

$x = 4$ cm

$h = \frac{144-3x^2}{4x} = 6$ cm

⑥



$V_{cylinder} = \pi r^2 h$

$SA_{cylinder} = 2\pi r^2 + 2\pi r h$

top and bottom   side

1) Minimize $A = 2\pi r^2 + 2\pi r h$

Restriction: $\pi r^2 h = 100$

2) Single variable

$$h = \frac{100}{\pi r^2}$$

$$f = 2\pi r^2 + \frac{2\pi r \cdot 100}{\pi r^2}$$

$$f = 2\pi r^2 + \frac{200}{r}$$

3) $f' = 0$

$$f' = 4\pi r - \frac{200}{r^2}$$

$$4\pi r - \frac{200}{r^2} = 0$$

$$4\pi r = \frac{200}{r^2}$$

$$r^3 = \frac{50}{\pi}$$

$$r = \sqrt[3]{\frac{50}{\pi}} \approx 2.52 \text{ cm}$$

Note: Could check using 2nd derivative test

$$f'' = 4\pi + \frac{400}{r^3}$$

$$f''\left(\sqrt[3]{\frac{50}{\pi}}\right) > 0 \Rightarrow f \text{ is concave up at } r = \sqrt[3]{\frac{50}{\pi}}$$

minimum ✓

4) Answer

$$r = \sqrt[3]{\frac{50}{\pi}} \approx 2.52 \text{ cm}$$

$$h = \frac{100}{\pi r^2} = \frac{100}{\pi \left(\frac{50}{\pi}\right)^{2/3}} \approx 5.03 \text{ cm}$$