24.2 Newton's Method

Ex: Find a root of $x^3+4x-18=0$ in the interval $[1,3]$

Newton's Method solves $f(x)=0$

x_0 : reasonable starting point

get x_1 : closer to solution

x_2 : even closer

$\vdots$

using $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

new x-value

old x-value

$\boxed{f(x) = x^3+4x-18}$

x	$f(x)$
1	-13
2	-2
3	21

Want $f(x)=0$

Choose x-value where $f(x)$ is closest to 0

$\boxed{x_0 = 2}$

$\boxed{f'(x) = 3x^2+4}$

x_n	$f(x_n)$	$f'(x_n)$	$x_n - \frac{f(x_n)}{f'(x_n)}$	
$x_0 = 2$	-2	16	2.125	(2.13)
$x_1 = 2.125$	0.0957	17.5469	2.1195	(2.12)
$x_2 = 2.1195$	-0.0006	17.4768	2.1195	(2.12)

Table to 4 decimal places
Stop when x-values agree to 2 d.p.
Solution to 2 d.p.

Solution $x \simeq 2.12$

Check: $2.12^3 + 4(2.12) - 18 \approx 0$ ✓

Ex: Approximate a solution to

$3\sqrt{x} + x^3 = 139$ in the interval $[4,6]$

$\rightarrow 3\sqrt{x} + x^3 - 139 = 0$

$$\boxed{f(x) = 3\sqrt{x} + x^3 - 139}$$

x	$f(x)$
4	-69
5	-7.3
6	84.3

Choose $x_0 = 5$

$$\boxed{f'(x) = \frac{3}{2}x^{-1/2} + 3x^2}$$

x_n	$f(x_n)$	$f'(x_n)$	$x_n - \frac{f(x_n)}{f'(x_n)}$	
5	-7.2918	75.6708	5.0964	(5.10)
5.0964	0.1429	78.5843	5.0946	(5.09)
5.0946	0.0015	78.5294	5.0946	(5.09)

Solution: $x \approx 5.09$ ✓

Ex: Find smallest root of $x^3+x^2-11x+7=0$

(furthest left on number line)

$$\boxed{f(x)=x^3+x^2-11x+7}$$

x	$f(x)$
-5	-38
-4	3
-3	22
-2	25
-1	18
0	7
1	-2
2	-3
3	1

Identify all 3 roots (where $f(x)$ changes sign) and locate smallest

Smallest root is in $[-5,-4]$

Choose $x_0=-4$

$$\boxed{f'(x)=3x^2+2x-11}$$

x_n	$f(x_n)$	$f'(x_n)$	$x_n-\frac{f(x_n)}{f'(x_n)}$
-4	3	29	-4.1034 (-4.10)
-4.1034	-0.1173	31.3069	-4.0997 (-4.10)

Solution: $x \simeq -4.10$