23.8 Implicit Differentiation

y is an <u>explicit</u> function of x means we have solved for y

e.g. $y = \frac{x-1}{x+1}$

Otherwise y is an <u>implicit</u> function of x

e.g. $x^2 + y^2 = 1$

$5x - 3x^2 y = \frac{4}{y}$

Can be difficult/impossible to solve for y

<u>Ex</u>: y is an implicit function of x
(y depends on x)

Find:

a) $\frac{d}{dx}[4x^2] = 8x$

b) $\frac{d}{dx}[4y^2] = 8y\frac{dy}{dx}$ Chain Rule

Multiply by $\frac{dy}{dx}$ when differentiating a y-term

c) $\frac{d}{dx}\left[x^3-2y^2+y\right] = 3x^2-4y\frac{dy}{dx}+\frac{dy}{dx}$

d) $\frac{d}{dx}\left[9xy\right] = \frac{d}{dx}\left[(9x)y\right]$

$= 9x\frac{dy}{dx} + y\cdot 9$ Product Rule

$= 9x\frac{dy}{dx} + 9y$

Main idea: Can find $\frac{dy}{dx}$ without solving for y first

Ex: Find $\frac{dy}{dx}$ @ $(x,y)=(0,2)$

for $5x-3x^2y+y^2=4$

1) Take $\frac{d}{dx}$ of both sides:

$5+\frac{d}{dx}\left[(-3x^2)y\right]+2y\frac{dy}{dx}=0$

$5-3x^2\frac{dy}{dx}-6xy+2y\frac{dy}{dx}=0$

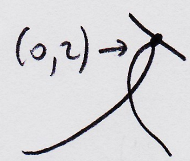
2) Solve for $\frac{dy}{dx}$

$$-3x^2\frac{dy}{dx} + 2y\frac{dy}{dx} = -5 + 6xy$$

$$\left(-3x^2 + 2y\right)\frac{dy}{dx} = -5 + 6xy$$

$$\frac{dy}{dx} = \frac{-5 + 6xy}{-3x^2 + 2y}$$

$$\left.\frac{dy}{dx}\right|_{(0,2)} = \frac{-5}{4}$$



<u>Ex</u>: Find $\frac{dy}{dx}$ given

$$4x^3y + (y^3 + x)^4 = 16$$

$$(4x^3)y + (y^3 + x)^4 = 16$$

1) Take $\frac{d}{dx}$

$$\underbrace{4x^3\frac{dy}{dx} + 12x^2y}_{\text{Product}} + \underbrace{4(y^3 + x)^3\left(3y^2\frac{dy}{dx}\right.}_{\text{Chain}}\left. + 1\right) = 0$$

2) Solve for $\frac{dy}{dx}$

Separate $\frac{dy}{dx}$ terms from non-$\frac{dy}{dx}$ terms

$$4x^3\frac{dy}{dx} + 12x^2y + 12y^2(y^3+x)^3\frac{dy}{dx} + 4(y^3+x)^3 = 0$$

$$4x^3\frac{dy}{dx} + 12y^2(y^3+x)^3\frac{dy}{dx} = -12x^2y - 4(y^3+x)^3$$

$$\left[4x^3 + 12y^2(y^3+x)^3\right]\frac{dy}{dx} = -12x^2y - 4(y^3+x)^3$$

$$\frac{dy}{dx} = \frac{-12x^2y - 4(y^3+x)^3}{4x^3 + 12y^2(y^3+x)^3}$$

<u>Ex</u>: Find $\frac{dy}{dx}$ given $x + \sqrt{x}\,y = \sqrt{y}$

$$x + x^{1/2}y = y^{1/2}$$

1) Take $\frac{d}{dx}$

$$1 + x^{1/2}\frac{dy}{dx} + \frac{1}{2}x^{-1/2}y = \frac{1}{2}y^{-1/2}\frac{dy}{dx}$$

2) Solve for $\frac{dy}{dx}$

$$\left(x^{1/2} - \frac{1}{2}y^{-1/2}\right)\frac{dy}{dx} = -1 - \frac{1}{2}x^{-1/2}y$$

$$\frac{dy}{dx} = \frac{-1 - \frac{1}{2}x^{-1/2}y}{x^{1/2} - \frac{1}{2}y^{-1/2}} \cdot \frac{2x^{1/2}y^{1/2}}{2x^{1/2}y^{1/2}}$$

Simplify : clear fractions and negative exponents

$$= \frac{-2x^{1/2}y^{1/2} - y^{3/2}}{2xy^{1/2} - x^{1/2}}$$

OR (mult. by $\frac{-1}{-1}$)

$$\frac{dy}{dx} = \frac{2x^{1/2}y^{1/2} + y^{3/2}}{-2xy^{1/2} + x^{1/2}}$$

Ex: Find $\frac{dy}{dx}$ given $x = \frac{x-y}{x+y}$

1) Take $\frac{d}{dx}$

$$1 = \frac{(x+y)(1-\frac{dy}{dx}) - (x-y)(1+\frac{dy}{dx})}{(x+y)^2}$$

2) Solve for $\frac{dy}{dx}$:

Mult. by $(x+y)^2$

$$(x+y)^2 = (x+y)(1-\frac{dy}{dx}) - (x-y)(1+\frac{dy}{dx})$$

Separate $\frac{dy}{dx}$ terms from non-$\frac{dy}{dx}$

$$(x+y)^2 = x+y - (x+y)\frac{dy}{dx} - (x-y) - (x-y)\frac{dy}{dx}$$

$$[(x+y)+(x-y)]\frac{dy}{dx} = x+y-(x-y)-(x+y)^2$$

$$2x\frac{dy}{dx} = 2y - (x+y)^2$$

$$\frac{dy}{dx} = \frac{2y-(x+y)^2}{2x}$$