23.7 General Power Rule and Chain Rule

Before : $\frac{d}{dx}[x^n] = nx^{n-1}$ for $n = 1, 2, 3, \ldots$

General Power Rule $\frac{d}{dx}[x^n] = nx^{n-1}$ n: any rational #
e.g. $n = \frac{2}{3}, -\frac{4}{5}, -7, 0, 3, \ldots$

Recap: exponents

$$\sqrt[3]{x} = x^{1/3}$$

$$\sqrt[7]{x^8} = x^{8/7}$$

$$\frac{1}{x^4} = x^{-4}$$

$$\frac{1}{\sqrt[7]{x^{11}}} = x^{-11/7}$$

Ex: Find $f'(x)$

a) $f(x) = \sqrt{x}$

$f(x) = x^{1/2}$

$f'(x) = \frac{1}{2}x^{-1/2}$ or $\frac{1}{2\sqrt{x}}$

b) $f(x) = \frac{1}{x}$

$f(x) = x^{-1}$

$f'(x) = -x^{-2}$ or $\frac{-1}{x^2}$

c) $f(x) = 5\sqrt[4]{x^3}$

$f(x) = 5x^{3/4}$

$f'(x) = \frac{15}{4}x^{-1/4}$ or $\frac{15}{4 \cdot \sqrt[4]{x}}$

d) $f(x) = \frac{12}{\sqrt{x^3}}$

$f(x) = 12x^{-3/2}$

$f'(x) = -18x^{-5/2}$

The Chain Rule

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$$ CHAIN RULE

derivative of outside function, keeping inside function the same

derivative of inside function

Ex: $y = (x^7 + 3x)^3$

$y' = 3(x^7 + 3x)^2 \cdot (7x^6 + 3)$

Alternative Version:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$ CHAIN RULE

Ex: $y = (x^7 + 3x)^3$

Let $y = u^3$ where $u = x^7 + 3x$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= 3u^2 (7x^6 + 3)$$

$$= 3(x^7 + 3x)^2 (7x^6 + 3)$$ same as above ✓

Ex: Find $\frac{dy}{dx}$

a) $y = 2(1-5x^4)^7$

$\frac{dy}{dx} = 14(1-5x^4)^6(-20x^3)$

$= -280x^3(1-5x^4)^6$

Leave answer factored

b) $y = 10x^3 - 6(1+5x)^4$

$\frac{dy}{dx} = 30x^2 - 24(1+5x)^3(5)$

$= 30x^2 - 120(1+5x)^3$

c) $y = \sqrt{2+4x^3}$

$y = (2+4x^3)^{1/2}$

$\frac{dy}{dx} = \frac{1}{2}(2+4x^3)^{-1/2}(12x^2)$

$= 6x^2(2+4x^3)^{-1/2}$

or $\frac{6x^2}{\sqrt{2+4x^3}}$

d) $y = \dfrac{1}{(1+17x^2)^4}$

$y = (1+17x^2)^{-4}$

$\dfrac{dy}{dx} = -4(1+17x^2)^{-5}(34x)$

$= \dfrac{-136x}{(1+17x^2)^5}$

<u>Ex</u>: Simplify y' where $y = 4x^2(8x+7)^6$

Product Rule $y' = 4x^2 \dfrac{d}{dx}\left[(8x+7)^6\right] + (8x+7)^6(8x)$

$y' = 4x^2\left[6(8x+7)^5(8)\right] + 8x(8x+7)^6$

Factor if possible

$y' = 8x(8x+7)^5\left[24x + (8x+7)\right]$

$= 8x(32x+7)(8x+7)^5$

Ex: Find $f'(1)$ for $f(x) = [(3x+2)(9x^2+1)]^4$

Chain Rule $f'(x) = 4[(3x+2)(9x^2+1)]^3 \frac{d}{dx}[(3x+2)(9x^2+1)]$

$= 4[(3x+2)(9x^2+1)]^3[(3x+2)(18x)+(9x^2+1)3]$

$f'(1) = 4[5(10)]^3[5(18)+10(3)]$

$= 6 \times 10^7$

Ex: $y = \frac{x}{\sqrt{1-3x}}$ Find y' and simplify.

Quotient Rule $y' = \frac{\sqrt{1-3x}\,(1) - x\frac{d}{dx}\sqrt{1-3x}}{\sqrt{1-3x}^{\,2}}$

$= \frac{\sqrt{1-3x} - x\left[\frac{1}{2}(1-3x)^{-1/2}(-3)\right]}{1-3x}$

simplify: clear fractions and negative exponents

$= \frac{\sqrt{1-3x} + \frac{3x}{2}(1-3x)^{-1/2}}{(1-3x)} \cdot \frac{2\sqrt{1-3x}}{2\sqrt{1-3x}}$

$= \frac{2(1-3x)+3x}{2(1-3x)^{3/2}} = \frac{2-3x}{2(1-3x)^{3/2}}$

$\underline{Ex}$: $f(x) = \sqrt{3x^2+6}\,(4-2x)$ Simplify $f'(x)$

Product Rule

$$f'(x) = \sqrt{3x^2+6}\,(-2) + (4-2x)\left[\tfrac{1}{2}(3x^2+6)^{-1/2}(6x)\right]$$

$$= -2\sqrt{3x^2+6} + \frac{3x(4-2x)}{\sqrt{3x^2+6}}$$ Common denominator

Mult. by $\frac{\sqrt{3x^2+6}}{\sqrt{3x^2+6}}$

$$= \frac{-2(3x^2+6)}{\sqrt{3x^2+6}} + \frac{3x(4-2x)}{\sqrt{3x^2+6}}$$

$$= \frac{-12x^2+12x-12}{\sqrt{3x^2+6}}$$

or $$\frac{-12(x^2-x+1)}{\sqrt{3x^2+6}}$$

Ex: $v = k\sqrt{\frac{l}{a} + \frac{a}{l}}$ k, a: constants $\neq 0$

l: length

Where is $\frac{dv}{dl} = 0$?

$$\frac{dv}{dl} = k\,\frac{1}{2}\left(\frac{l}{a} + al^{-1}\right)^{-1/2}\left(\frac{1}{a} - al^{-2}\right)$$

$$= \frac{k\left(\frac{1}{a} - al^{-2}\right)}{2\sqrt{\frac{l}{a} + \frac{a}{l}}}$$

Set $\frac{dv}{dl} = 0$

$$\frac{k\left(\frac{1}{a} - al^{-2}\right)}{2\sqrt{\frac{l}{a} + \frac{a}{l}}} = 0$$

$$k\left(\frac{1}{a} - al^{-2}\right) = 0$$

$$\frac{1}{a} - al^{-2} = 0$$

$$\frac{1}{a} = \frac{a}{l^2}$$

$$l^2 = a^2$$

$$l = \pm a$$

$l > 0$ because it's a length $\Rightarrow$ $\boxed{l = a}$