23.3 The Derivative

The derivative of $f(x)$ is written $f'(x)$

$f'(x)$ represents: slope of tangent line

instantaneous rate of change of $f(x)$

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Differentiate $f(x)$: find $f'(x)$

Ex: $f(x) = 2x^3 - \pi$ Find $f'(x)$ and $f'(-1)$

Hint: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{2(x+h)^3 - \pi - [2x^3 - \pi]}{h}$$

$$= \lim_{h \to 0} \frac{2(x^3 + 3x^2h + 3xh^2 + h^3) - \pi - 2x^3 + \pi}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2x^3} + 6x^2h + 6xh^2 + 2h^3 - \cancel{2x^3}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(6x^2 + 6xh + 2h^2)}{\cancel{h}}$$

$$= 6x^2$$

$f'(x) = 6x^2$

$f'(-1) = 6$

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f(x) is differentiable : means $f'(x)$ is defined

$f'(x) = 6x^2$ is defined for all x

$f(x) = 2x^3 - \pi$ is differentiable for all x

Recap: Common denominators

$$\frac{2}{x+h-1} - \frac{2}{x-1}$$

$$= \frac{2(x-1) - 2(x+h-1)}{(x+h-1)(x-1)}$$

$$= \frac{2x - 2 - 2x - 2h + 2}{(x+h-1)(x-1)}$$

expand numerator
keep denom. factored

$$= \frac{-2h}{(x+h-1)(x-1)}$$

Ex: $f(x) = x^2 + \frac{4}{x}$ Find $f'(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[(x+h)^2 + \frac{4}{x+h} - \left(x^2 + \frac{4}{x}\right)\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\cancel{x^2} + 2xh + h^2 + \frac{4}{x+h} - \cancel{x^2} - \frac{4}{x}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[2xh + h^2 + \frac{4x}{x(x+h)} - \frac{4(x+h)}{x(x+h)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[2xh + h^2 - \frac{4h}{x(x+h)}\right]$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}}\left[2x + h - \frac{4}{x(x+h)}\right]$$

$$= 2x - \frac{4}{x^2}$$

$f'(x)$ is defined for all $x \neq 0$

$f(x)$ is differentiable for all $x \neq 0$

Ex: $\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$ $\left(\frac{0}{0}\right)$ multiply top and bottom by conjugate radical

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h}\sqrt{x+h} + \cancel{\sqrt{x+h}\sqrt{x}} - \cancel{\sqrt{x}\sqrt{x+h}} - \sqrt{x}\sqrt{x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{x+h - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}\ 1}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \frac{1}{2\sqrt{x}}$$

Ex: $f(x) = \sqrt{x+3}$ Find $f'(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h+3} - \sqrt{x+3}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h+3} - \sqrt{x+3}}{h} \cdot \frac{\sqrt{x+h+3} + \sqrt{x+3}}{\sqrt{x+h+3} + \sqrt{x+3}}$$

$$= \lim_{h \to 0} \frac{x+h+3 - (x+3)}{h\left(\sqrt{x+h+3} + \sqrt{x+3}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}\ 1}{\cancel{h}\left(\sqrt{x+h+3} + \sqrt{x+3}\right)}$$

$$= \frac{1}{2\sqrt{x+3}}$$

$f'(x)$ exists for $x > -3$

$f(x)$ is differentiable for $x > -3$

Notation:

$f(x) = x^2$ or $y = x^2$

The derivative $f'(x) = 2x$

or $y' = 2x$

or $\frac{df}{dx} = 2x$ ← "derivative of f with respect to x"

or $\frac{dy}{dx} = 2x$

or $\frac{d}{dx}[x^2] = 2x$

$f'(1) = 2$

$y'\big|_{x=1} = 2$

$\frac{df}{dx}\Big|_{x=1} = 2$

$\frac{dy}{dx}\Big|_{x=1} = 2$