16.5 Gauss-Jordan Elimination

$$\begin{cases} x + 2y = 7 \\ 3x + 4y = 15 \end{cases}$$

will be written

$$\begin{array}{ccc} x & y & \# \\ \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 3 & 4 & 15 \end{array}\right]$$

A matrix is in <u>reduced row echelon form</u> (RREF) if:

1) First nonzero entry in each <u>row</u> is 1
2) These leading 1's have 0 everywhere else in their <u>columns</u>
3) Leading 1's move down and right
4) Any zero rows are at the bottom

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 1 & \end{array}\right]$$

RREF ✓

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & \end{array}\right]$$

RREF ✓

$$\left[\begin{array}{ccc|c} 0 & 1 & 5 & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \end{array}\right]$$

RREF ✓

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & \\ 0 & 1 & 0 & \\ 0 & 0 & 0 & \end{array}\right]$$

not RREF

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & \\ 1 & 0 & 0 & \\ 0 & 0 & 0 & \end{array}\right]$$

not RREF

Ex: Solve using Gauss-Jordan Elimination

$$\begin{cases} 3x - 4y + z = 25 \\ 2x + 4y + z = -16 \\ x \quad\quad + 5z = 11 \end{cases}$$

$$\begin{array}{cccc} x & y & z & \# \\ \multicolumn{4}{c}{\left[\begin{array}{ccc|c} 3 & -4 & 1 & 25 \\ 2 & 4 & 1 & -16 \\ 1 & 0 & 5 & 11 \end{array}\right]} \end{array}$$

Gauss-Jordan Elimination:

left side → RREF using 3 operations

1) Swap 2 rows
2) Mult/divide a row by a nonzero #
3) Replace R_i with $R_i \pm kR_j$

   e.g. $R_2 \to R_2 - 4R_1$

Get a 1

$R_1 \leftrightarrow R_3$ $\left[\begin{array}{ccc|c} 1 & 0 & 5 & 11 \\ 2 & 4 & 1 & -16 \\ 3 & -4 & 1 & 25 \end{array}\right]$

Get 0's

$R_2 - 2R_1$
$R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 11 \\ 0 & 4 & -9 & -38 \\ 0 & -4 & -14 & -8 \end{array}\right]$$

Get a 1
$R_2/4$

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 11 \\ 0 & 1 & -\frac{9}{4} & -\frac{38}{4} \\ 0 & -4 & -14 & -8 \end{array}\right]$$

Get 0's

$R_3 + 4R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 11 \\ 0 & 1 & -\frac{9}{4} & -\frac{38}{4} \\ 0 & 0 & -23 & -46 \end{array}\right]$$

Get a 1

$R_3/(-23)$

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 11 \\ 0 & 1 & -\frac{9}{4} & -\frac{38}{4} \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$R_1 - 5R_3$
$R_2 + \frac{9}{4}R_3$

$$\begin{array}{cccc} x & y & z & \# \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{array}\right] \leftarrow -\frac{38}{4} + \frac{9}{4}(2) = -5$$

$x = 1$
$y = -5$
$z = 2$

$(x, y, z) = (1, -5, 2)$

$\underline{Ex}$: Solve $\begin{cases} x + 2y + 2z = 4 \\ 3x + y + 4z = 6 \\ \quad 4y + 2z = 5 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 3 & 1 & 4 & 6 \\ 0 & 4 & 2 & 5 \end{array}\right]$$

Get 0's

$R_2 - 3R_1$ $$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 0 & -5 & -2 & -6 \\ 0 & 4 & 2 & 5 \end{array}\right]$$

Get a 1

$R_2/(-5)$ $$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 0 & 1 & \frac{2}{5} & \frac{6}{5} \\ 0 & 4 & 2 & 5 \end{array}\right]$$

$R_1 - 2R_2$

$R_3 - 4R_2$ $$\left[\begin{array}{ccc|c} 1 & 0 & \frac{6}{5} & \frac{8}{5} \\ 0 & 1 & \frac{2}{5} & \frac{6}{5} \\ 0 & 0 & \frac{2}{5} & \frac{1}{5} \end{array}\right]$$

$R_3 \times \frac{5}{2}$ $$\left[\begin{array}{ccc|c} 1 & 0 & \frac{6}{5} & \frac{8}{5} \\ 0 & 1 & \frac{2}{5} & \frac{6}{5} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

$R_1 - \frac{6}{5}R_3$

$R_2 - \frac{2}{5}R_3$ $$\begin{array}{cccc} x & y & z & \# \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right] \begin{array}{l} \leftarrow \frac{8}{5} - \frac{6}{5}\left(\frac{1}{2}\right) \\ \leftarrow \frac{6}{5} - \frac{2}{5}\left(\frac{1}{2}\right) \\ \end{array}$$

$$\boxed{(x, y, z) = \left(1, 1, \tfrac{1}{2}\right)}$$

Ex: Solve

$$\begin{array}{cccc} x & y & z & \# \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 1 & 3 & 3 & 12 \\ 1 & 4 & 5 & 1 \end{array}\right]$$

$$\begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & -8 \end{array}\right]$$

$$\begin{array}{l} R_1 - 2R_2 \\ \\ R_3 - 2R_2 \end{array} \quad \left[\begin{array}{ccc|c} & & & \\ & & & \\ 0 & 0 & 0 & -14 \end{array}\right]$$

$$0 = -14$$

impossible

" System has no solution.
System is inconsistent "

Ex: If you see a row like this, system has no solution:

$$[\,0 \quad 0 \mid 3\,]$$

$$[\,0 \quad 0 \quad 0 \mid -2\,]$$

[all zero | nonzero]

<u>Ex</u>: Solve

$$\begin{array}{cccc} x & y & z & \# \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 1 & 3 & 3 & 12 \\ 1 & 4 & 5 & 15 \end{array}\right]$$

$$\begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 9 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \end{array}\right]$$

$$\begin{array}{cccc} x & y & z & \# \end{array}$$
$$\begin{array}{l} R_1 - 2R_2 \\ \\ R_3 - 2R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

RREF

$$\begin{array}{cccc} x & y & z & \# \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\uparrow$

Any column without a leading 1 gets a <u>parameter</u>

(∞-many solutions)

$\boxed{z = t} \leftarrow$ any real #

$y + 2z = 3$

$y = 3 - 2z$

$\boxed{y = 3 - 2t}$

$x - 3z = 3$

$x = 3 + 3z$

$\boxed{x = 3 + 3t}$

$(x,y,z) = (3+3t,\ 3-2t,\ t)$

Some particular solutions:

$(t=0) \qquad (x,y,z) = (3,3,0)$

$(t=1) \qquad (x,y,z) = (6,1,1)$

$(t=-1) \qquad (x,y,z) = (0,5,-1)$

etc.

Most triples are not solutions

e.g. $(1,1,1)$ is not a solution.

<u>Ex</u>: Solve

$$\begin{array}{cccc} x & y & z & \# \end{array}$$
$$\left[\begin{array}{ccc|c} 4 & -8 & 12 & 20 \\ 3 & 1 & 1 & 5 \end{array}\right]$$

$$R_1/4 \quad \left[\begin{array}{ccc|c} 1 & -2 & 3 & 5 \\ 3 & 1 & 1 & 5 \end{array}\right]$$

$$R_2-3R_1 \quad \left[\begin{array}{ccc|c} 1 & -2 & 3 & 5 \\ 0 & 7 & -8 & -10 \end{array}\right]$$

$$R_2/7 \quad \left[\begin{array}{ccc|c} 1 & -2 & 3 & 5 \\ 0 & 1 & -\frac{8}{7} & -\frac{10}{7} \end{array}\right]$$

$R_1 + 2R_2$

$$\begin{array}{c} \begin{array}{cccc} x & y & z & \# \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & \frac{5}{7} & \frac{15}{7} \\ 0 & 1 & -\frac{8}{7} & -\frac{10}{7} \end{array}\right] \end{array}$$

RREF

(∞-many solutions)

$$\boxed{z = t}$$

$$y - \frac{8}{7}z = \frac{-10}{7}$$

$$y = \frac{-10}{7} + \frac{8}{7}z$$

$$\boxed{y = \frac{-10}{7} + \frac{8}{7}t}$$

$$x + \frac{5}{7}z = \frac{15}{7}$$

$$x = \frac{15}{7} - \frac{5}{7}z$$

$$\boxed{x = \frac{15}{7} - \frac{5}{7}t}$$

$$(x, y, t) = \left(\frac{15}{7} - \frac{5}{7}t, \frac{-10}{7} + \frac{8}{7}t, t\right)$$

Ex: Solve

$$\begin{array}{c} \begin{array}{cccc} x & y & z & \# \end{array} \\ \left[\begin{array}{ccc|c} 3 & 6 & 6 & 12 \\ 3 & 6 & 8 & 15 \\ 2 & 4 & 4 & 8 \end{array}\right] \end{array}$$

$R_1/3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 3 & 6 & 8 & 15 \\ 2 & 4 & 4 & 8 \end{array}\right]$$

$R_2 - 3R_1$
$R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Can't make this 1
Move further right

$R_2/2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 - 2R_2$

$$\begin{array}{c} \begin{array}{cccc} x & y & z & \# \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

(∞-many solutions)

$\boxed{y = t}$

$\boxed{z = 3/2}$

$x + 2y = 1$
$x = 1 - 2y$
$\boxed{x = 1 - 2t}$

$(x, y, z) = (1-2t,\ t,\ \frac{3}{2})$

Recap: A system can have 0, 1 or ∞-many solutions

1) 1 solution

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$x = 4$
$y = 5$
$z = 6$

2) No solution

$$\left[\begin{array}{ccc|c} & & & \\ 0 & 0 & 0 & 7 \end{array}\right]$$

3) ∞-many solutions

$$\begin{array}{ccc|c} x & y & z & \# \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 4 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\boxed{z = t}$

$y + 6z = 3$

$\boxed{y = 3 - 6t}$

$x - 2z = 4$

$\boxed{x = 4 + 2t}$

Two Methods for Solving a System:

1) $AX = B$

   $X = A^{-1}B$

   - Fast
   - Only works when A^{-1} exists

2) Gauss-Jordan Elimination
   - More work
   - More powerful