

16.4 Matrices and Systems of Equations

$$\begin{cases} 2x + 3y = -4 \\ 3x + 5y = -5 \end{cases} \quad \text{system of equations}$$

as a matrix equation :

$$\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \end{bmatrix}$$

more generally : $AX = B$

(saw details in Section 16.2)

A system of equations $AX = B$ has
solution $X = A^{-1}B$ (if A^{-1} exists)

Why?

$$AX = B$$

$$\underbrace{A^{-1}}_I AX = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B \quad \checkmark$$

Ex: Solve by finding A^{-1}

$$\begin{cases} 2x + 3y = -4 \\ 3x + 5y = -5 \end{cases}$$

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$$

$$\det A = 2(5) - 3(3) = 1$$

$$A^{-1} = \frac{1}{1} \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$$

$$X = A^{-1}B$$

$$= \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -4 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

$$\text{i.e. } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

$$\text{Check: } 2(-5) + 3(2) = -4 \quad \checkmark$$

$$3(-5) + 5(2) = -5 \quad \checkmark$$

Ex: Solve by finding A^{-1}

$$\begin{cases} 7x - 3y = 25 \\ 2x + y = 9 \end{cases}$$

$$A = \begin{bmatrix} 7 & -3 \\ 2 & 1 \end{bmatrix}$$

$$\det A = 7(1) - (-3)(2) = 13$$

$$A^{-1} = \frac{1}{13} \begin{bmatrix} 1 & 3 \\ -2 & 7 \end{bmatrix}$$

$$X = A^{-1}B$$

$$= \frac{1}{13} \begin{bmatrix} 1 & 3 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} 25 \\ 9 \end{bmatrix}$$

$$= \frac{1}{13} \begin{bmatrix} 52 \\ 13 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad \checkmark$$

Ex: Solve by finding A^{-1}

$$\begin{cases} x + 2y + 2z = 17 \\ x + y + z = 9 \\ x - 2z = -9 \end{cases}$$

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{bmatrix}$$

$$[A \mid I] \rightsquigarrow [I \mid A^{-1}]$$

$$\left[\begin{array}{ccc|ccc} \textcircled{1} & 2 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 & 0 & 1 \end{array} \right]$$

$$\begin{array}{l} R_2 - R_1 \\ R_3 - R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & -2 & -4 & -1 & 0 & 1 \end{array} \right]$$

$$R_2 / (-1) \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & \textcircled{1} & 1 & 1 & -1 & 0 \\ 0 & -2 & -4 & -1 & 0 & 1 \end{array} \right]$$

$$\begin{array}{l} R_1 - 2R_2 \\ R_3 + 2R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & -2 & 1 & -2 & 1 \end{array} \right]$$

$$R_3/(-2) \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & \textcircled{1} & -\frac{1}{2} & 1 & -\frac{1}{2} \end{array} \right]$$

$$R_2 - R_3 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & 1 & -\frac{1}{2} \end{array} \right]$$

A^{-1}

$$X = A^{-1} B$$

$$= \begin{bmatrix} -1 & 2 & 0 \\ \frac{3}{2} & -2 & \frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 17 \\ 9 \\ -9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$