16.3 The Inverse of a Matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the determinant of A

is $\det A = ad - bc$.

$$\boxed{A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

Notes: 1) If $\det A = 0$ then A^{-1} does not exist

2) Formula only works for 2x2 matrices

Ex: Find A^{-1}

a) $A = \begin{bmatrix} 9 & 6 \\ 2 & 1 \end{bmatrix}$

$\det A = -3$

$$A^{-1} = \frac{1}{-3}\begin{bmatrix} 1 & -6 \\ -2 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{3} & 2 \\ \frac{2}{3} & -3 \end{bmatrix}$$

Check: $A^{-1}A = \begin{bmatrix} -\frac{1}{3} & 2 \\ \frac{2}{3} & -3 \end{bmatrix}\begin{bmatrix} 9 & 6 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

b) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

det $A = 0$

A^{-1} does not exist

Gauss-Jordan Method for finding A^{-1}

$[A \mid I] \rightsquigarrow [I \mid A^{-1}]$ using 3 operations:

1) Swap 2 rows
2) Multiply/divide a row by any nonzero #
3) Replace R_i with $R_i \pm kR_j$ e.g. $R_2 \to R_2 - 3R_1$

Notes: 1) If we create a row of 0's then A^{-1} does not exist
2) Works for 2x2, 3x3, 4x4 etc.

Ex: Find A^{-1} for $A = \begin{bmatrix} 9 & 6 \\ 2 & 1 \end{bmatrix}$

$[A \mid I]$

$$\left[\begin{array}{cc|cc} 9 & 6 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right]$$

$R_1/9$ $\left[\begin{array}{cc|cc} 1 & 2/3 & 1/9 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right]$ $6/9 = 2/3$

OMIT

$$R_2 - 2R_1 \left[\begin{array}{cc|cc} 1 & 2/3 & 1/9 & 0 \\ 0 & \frac{-1}{3} & \frac{-2}{9} & 1 \end{array}\right]$$

$1 - 2\left(\frac{2}{3}\right)$

OMIT

$$R_2 \times (-3) \left[\begin{array}{cc|cc} 1 & 2/3 & 1/9 & 0 \\ 0 & 1 & 2/3 & -3 \end{array}\right]$$

$$R_1 - \frac{2}{3}R_2 \left[\begin{array}{cc|cc} 1 & 0 & -1/3 & 2 \\ 0 & 1 & 2/3 & -3 \end{array}\right]$$

$\frac{1}{9} - \frac{2}{3}\left(\frac{2}{3}\right) = -3/9$

$[\, I \mid A^{-1} \,]$

$$A^{-1} = \begin{bmatrix} \frac{-1}{3} & 2 \\ 2/3 & -3 \end{bmatrix}$$

Check: $A^{-1}A = I$ ✓

Ex: Find A^{-1} $\quad A = \begin{bmatrix} 2 & 10 & 2 \\ 0 & 4 & 1 \\ 2 & 14 & 2 \end{bmatrix}$

$[\, A \mid I \,]$

$$\left[\begin{array}{ccc|ccc} 2 & 10 & 2 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 & 1 & 0 \\ 2 & 14 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$R_1/2 \quad \left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1/2 & 0 & 0 \\ 0 & 4 & 1 & 0 & 1 & 0 \\ 2 & 14 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$R_3 - 2R_1 \quad \left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1/2 & 0 & 0 \\ 0 & 4 & 1 & 0 & 1 & 0 \\ 0 & 4 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$R_2/4 \quad \left[\begin{array}{ccc|ccc} 1 & 5 & 1 & 1/2 & 0 & 0 \\ 0 & 1 & 1/4 & 0 & 1/4 & 0 \\ 0 & 4 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_1 - 5R_2 \\ \\ R_3 - 4R_2 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -1/4 & 1/2 & -5/4 & 0 \\ 0 & 1 & 1/4 & 0 & 1/4 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right]$$

$$R_3/(-1) \quad \left[\begin{array}{ccc|ccc} 1 & 0 & -1/4 & 1/2 & -5/4 & 0 \\ 0 & 1 & 1/4 & 0 & 1/4 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

$$R_1 + \frac{1}{4}R_3,\quad R_2 - \frac{1}{4}R_3 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -1 & -\frac{1}{4} \\ 0 & 1 & 0 & -\frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

$[I \mid A^{-1}]$

$\uparrow$ A^{-1}

To check: $A^{-1}A = I$ ✓

<u>Ex</u>: Find A^{-1} for $A = \begin{bmatrix} 1 & 1 & 6 \\ 0 & 2 & 3 \\ 2 & 2 & 12 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 6 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 2 & 2 & 12 & 0 & 0 & 1 \end{array}\right]$$

$$R_3 - 2R_1 \quad \left[\begin{array}{ccc|ccc} & & & & & \\ & & & & & \\ 0 & 0 & 0 & -2 & 0 & 1 \end{array}\right]$$

row of 0's

A^{-1} does not exist

<u>Ex</u>: Find A^{-1} for $A = \begin{bmatrix} 5 & 6 & 7 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 5 & 6 & 7 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \leftrightarrow R_3$ $$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 5 & 6 & 7 & 1 & 0 & 0 \end{array}\right]$$

$R_2 - R_1$
$R_3 - 5R_1$ $$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 6 & 2 & 1 & 0 & -5 \end{array}\right]$$

$R_3 - 6R_2$ $$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 & -6 & 1 \end{array}\right]$$

$R_3/2$ $$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & -3 & \frac{1}{2} \end{array}\right]$$

$R_1 - R_3$ $$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-1}{2} & 3 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & -3 & \frac{1}{2} \end{array}\right]$$

" A^{-1}

To check: $A^{-1}A = \underline{I}$ ✓