16.2 Matrix Multiplication

Dot product $\begin{bmatrix} 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 5 \end{bmatrix} = 1(1) + 3(5) = 16$

$$\begin{bmatrix} 1 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 6 \\ 2 \end{bmatrix} = 1(-1) + 4(6) + 1(2)$$

$$= 25$$

<u>Ex</u>: $A = \begin{bmatrix} 4 & 6 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$

rows of A
Columns of B

$$AB = \begin{bmatrix} r_1 \cdot c_1 & r_1 \cdot c_2 \\ r_2 \cdot c_1 & r_2 \cdot c_2 \\ r_3 \cdot c_1 & r_3 \cdot c_2 \end{bmatrix}$$

$4(1) + 6(1) + 1(2)$

$4(0) + 6(1) + 1(3)$

$$= \begin{bmatrix} 12 & 9 \\ 8 & 11 \\ 4 & 4 \end{bmatrix}$$

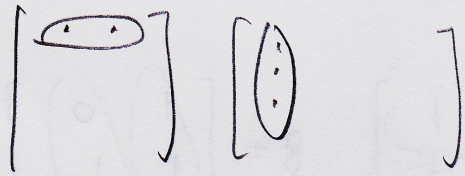
Size of AB: 3×2

$$\underbrace{\overset{A}{(3 \times \underbrace{3)} \overset{B}{(3}}_{\text{must be equal}} \times 2)}_{\text{Size of AB}}$$

BA is undefined :

$$\underset{(3 \times 2)}{B} \quad \underset{(3 \times 3)}{A}$$

not equal



mismatch

$$\boxed{\text{Caution: } AB \neq BA \text{ in general}}$$

<u>Ex</u>: Find CD and DC, if possible

$$C = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 1 & 1 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$$

CD is undefined

$$DC = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 6 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & 5 & 9 & 17 \end{bmatrix}$$

Ex: $A = \begin{bmatrix} 1 & 9 \\ -4 & 3 \end{bmatrix}$ Find A^2

$$A^2 = AA$$

$$= \begin{bmatrix} 1 & 9 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 9 \\ -4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -35 & 36 \\ -16 & -27 \end{bmatrix}$$

Why do we multiply like this?

Let $A = \begin{bmatrix} 1 & -4 \\ 2 & 3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \end{bmatrix}$ $B = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$

Matrix equation $AX = B$

$$\begin{bmatrix} 1 & -4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x - 4y \\ 2x + 3y \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

$$\begin{cases} x - 4y = 7 & (1) \\ 2x + 3y = 6 & (2) \end{cases}$$ system of equations

Ex: Write as a matrix equation

$$\begin{cases} 2x - 9y = 12 \\ 7x + 3y = -16 \end{cases}$$

$$\begin{bmatrix} 2 & -9 \\ 7 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 \\ -16 \end{bmatrix}$$

Coefficients → variables (column) constants (column)

Identity matrices

(2x2) $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(3x3) $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(4x4) $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

etc.

$\boxed{IA = A \text{ for any matrix } A}$

Ex: Confirm that $IA = A$ for
$A = \begin{bmatrix} 9 & 8 \\ 2 & 3 \end{bmatrix}$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 9 & 8 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 \\ 2 & 3 \end{bmatrix}$$

$$= A \checkmark$$

For a square matrix A
(2×2, 3×3 etc.)

A^{-1} has the property that:

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I$$

A^{-1} is called the <u>inverse of A</u>.

<u>Note</u>: Some square matrices don't have an inverse.

<u>Ex</u>: $A = \begin{bmatrix} 1 & -4 \\ -2 & 9 \end{bmatrix}$

Check that $\begin{bmatrix} 9 & 4 \\ 2 & 1 \end{bmatrix} = A^{-1}$

→ Check $AA^{-1} = I$ or $A^{-1}A = I$
Only need to check one.

$$AA^{-1} = \begin{bmatrix} 1 & -4 \\ -2 & 9 \end{bmatrix} \begin{bmatrix} 9 & 4 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I \quad \checkmark$$

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ doesn't have an inverse.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} w & x \\ 0 & 0 \end{bmatrix}$$

cannot equal $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

No A^{-1} exists so that $AA^{-1} = I$

Preview of 16.4

Want to solve $AX = B$ ← e.g. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

$$\underbrace{A^{-1}A}_{I} X = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \# \\ \# \end{bmatrix}$$

fast way to solve